Chapter

8

Discrete Functions: Financial Applications

▶ GOALS

You will be able to

- Determine how interest is earned and charged
- Use the difference between future value and present value to solve problems
- Solve problems about money invested at regular intervals over a period of time
- Calculate payments that must be made when a purchase is financed over a period of time

? On your first birthday, your parents deposit $1000 into a bank account that pays 3% interest per year on the balance. How can you determine which amount will be closest to what will be in the account when you are ready to go to college or university, $1100, $1500, $2000, or $2500?

8 Getting Started

SKILLS AND CONCEPTS You Need

Study Aid

For help, see the following lessons in Chapters 4 and 7.

Question	Lesson
1, 2, 3	7.1 and 7.2
4	7.5 and 7.6
5, 6	4.7

1. For each sequence, determine
  i) the next two terms **ii)** the general term **iii)** the recursive formula

  a) 7, 11, 15, 19, ...
  b) 58, 31, 4, −23, ...
  c) 5, 20, 80, 320, ...
  d) 1000, −500, 250, ...

2. The fourth term of an arithmetic sequence is 46 and the sixth term is 248. Determine
  a) the 5th term
  b) the common difference
  c) the 1st term
  d) the 100th term

3. The 4th, 5th, and 6th terms of a sequence are 9261, 9724.05, and 10 210.2525, respectively.
  a) What type of sequence is this? Justify your reasoning.
  b) Determine the recursive formula.
  c) State the general term.
  d) Determine the 10th term.

4. Determine the sum of the first 10 terms of each series.
  a) $3 + 5 + 7 + \ldots$
  b) $-27 - 21 - 15 - \ldots$
  c) $48 + 31 + 14 + \ldots$
  d) $8\,192\,000 - 4\,096\,000 + 2\,048\,000 - \ldots$

5. The population of a city is 200 000 and increases by 5% per year.
  a) Determine the expected population at the end of each of the next 3 years.
  b) What will be the expected population 10 years from now?

6. Determine the value of x that makes the equation $2^x = 4096$ true.

7. Solve each equation by graphing the corresponding functions on a graphing calculator. Round your answers to two decimal places.
  a) $2^x = 1\,000\,000$
  b) $5 \times 3^x = 228$
  c) $14\,000 \times 1.07^x = 30\,000$
  d) $250 \times 1.0045^{12x} = 400$

8. Complete the chart shown by writing what you know about exponential functions.

Example:	Visual representation:
Definition in your own words:	Personal association:

APPLYING What You Know

YOU WILL NEED

- graphing calculator
- spreadsheet software (optional)

Saving for a Trip

Mark is saving for an overseas trip that costs $1895. Each week, he deposits $50 into a savings account that pays him 0.35% on the minimum monthly balance. He already has $200 in his account at the start of the month.

? Assuming that the cost of the trip stays the same, how long will it take Mark to save enough money to pay for it?

A. How do you know it will take less than three years for Mark to meet his goal?

B. When Mark pays for the trip, he will have to pay the Goods and Services Tax (GST). Determine the additional costs he will incur. Calculate the total price of the trip.

C. What do you think earning 0.35% on the minimum monthly balance means? How will earning interest affect the amount of time he will have to save?

D. Determine how much Mark will have in his account at the end of the 1st, 2nd, and 3rd months. Record your values in a table as shown.

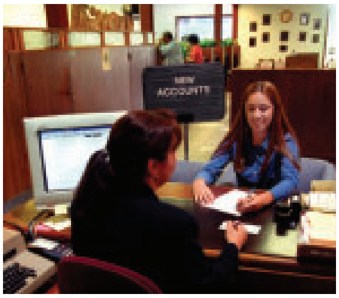
Month	Starting Balance	Deposits	Interest Earned	Final Balance
1				
2				
3				

E. Use a spreadsheet or the lists on a graphing calculator, or continue the table in part D to determine how long it will take Mark to save for the trip.

8.1 Simple Interest

YOU WILL NEED

- graphing calculator
- spreadsheet software



principal
a sum of money that is borrowed or invested

simple interest
interest earned or paid only on the original sum of money that was invested or borrowed

interest
the money earned from an investment or the cost of borrowing money

amount
the total value of an investment or loan. The amount is given by $A = P + I$, where A is the amount, P is the principal, and I is the interest.

Tech Support

For help using a graphing calculator to enter lists and to create scatter plots, see Technical Appendix, B-11.

GOAL

Calculate simple interest.

INVESTIGATE the Math

Amanda wants to invest $2000. Her bank will pay 6% of the **principal** per year each year the money is kept in a savings account that earns **simple interest**.

? What function can be used to model the growth of Amanda's money?

A. Calculate the **interest** earned and the **amount** of the investment at the end of the first year. Record your results in a table as shown.

Year	Interest Earned	Amount
0	—	$2000
1		
2		
3		

B. Calculate the interest earned and the amount of the investment at the end of the second and third years. Record your results in your table.

C. Enter your data for Year and Amount into either lists on a graphing calculator or columns in a spreadsheet.

D. Create a scatter plot from your two lists or columns, using Year as the independent variable.

E. What type of function best models the growth of Amanda's money? You may need to calculate more data points before you decide. Explain your reasoning.

F. Determine the equation of the function that models the amount of Amanda's investment over time.

Reflecting

G. What type of sequence could you use to represent the amount of Amanda's investment for successive years? How do you know?

H. How does the recursive formula for this sequence relate to her investment?

I. How do the principal, interest, and amount of Amanda's investment relate to
- the sequence from part G that represents the amount of the investment over time?
- the function from part F that represents the amount of the investment over time?

APPLY the Math

EXAMPLE 1 Representing any situation earning simple interest as a function

Allen invests \$3240 at 2.4%/a simple interest.

a) Calculate the interest earned each year.

b) Calculate the amount and the total interest earned after 20 years.

c) Determine the total amount, A, and the interest, I, earned if he invested a principal, $\$P$, for t years at r%/a simple interest.

Communication *Tip*

Interest rates are often advertised as a certain percent per year. So a rate of 5% means that 5% interest is earned each year. These rates are sometimes abbreviated to 5%/a, which means 5% per *annum*, or year.

Jasmine's Solution

a) $I = 0.024 \times \$3240$

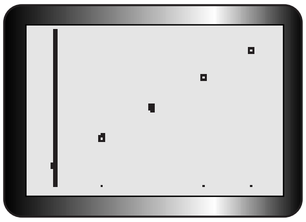
$= \$77.76$

Each year, Allen earns 2.4% of the principal as interest. 2.4% as a decimal is 0.024. To calculate the interest earned each year, I multiplied the principal by the interest rate.

b)

Year	Interest (\$)	Amount (\$)
0	0	3240
1	77.76	3240 + 77.76 = 3317.76
2	77.76	3317.76 + 77.76 = 3395.52
3	77.76	3395.52 + 77.76 = 3473.28
4	77.76	3473.28 + 77.76 = 3551.04

I set up a table to calculate the amount at the end of each year. I added the interest earned each year to the previous amount. Then I entered the year and amount into separate lists on a graphing calculator.



I used the calculator to create a scatter plot of amount versus time. The graph is linear. I used two points to calculate the slope and I found that it was \$77.76. That is the rate of change of the amount. The *y*-intercept is Allen's principal of \$3240.

$$f(t) = 77.76t + 3240$$

I used the slope and *y*-intercept to create a linear function in terms of *t*, the time in years, and $f(t)$, the amount.

$$f(20) = 77.76(20) + 3240$$
$$= 4795.20$$

To determine the amount after 20 years, I substituted $t = 20$.

$$I = 4795.20 - 3240$$
$$= 1555.20$$

To determine the total interest earned, I subtracted the principal from the amount.

After 20 years, Allen will have \$4795.20 and will have earned \$1555.20 in interest.

c) end of 1st year:

$$I_1 = Pr$$

After one year, Allen would earn *r*% of his original investment of \$*P*.

end of 2nd year:

$$I_2 = Pr + Pr = 2Pr$$

Each year, the total interest earned would go up by the same amount. The increase would be the interest earned in one year, $P \times r$.

end of 3rd year:

$$I_3 = Pr + Pr + Pr = 3Pr$$

end of *t*th year:

$$I = Prt$$

Since the interest earned at the end of each year depends on time, I wrote a formula for interest in terms of *t*, the time in years.

$$A = P + I$$
$$A = P + Prt$$

The amount is the sum of the principal and the interest. Since interest depends on time, the amount must also depend on time.

$$A = P(1 + rt)$$

I wrote the formula for the amount by factoring out the *P*.

The total amount of an investment of \$*P* for *t* years at *r*%/a simple interest is $A = P(1 + rt)$, and the total interest earned is $I = Prt$.

EXAMPLE 2 Using a spreadsheet to represent the amount owed

Tina borrows $15 000 at 6.8%/a simple interest. She plans to pay back the loan in 10 years. Calculate how much she will owe at the end of each year during this period.

Tom's Solution

	A	B	C
1	Time (Years)	Total Interest Charged	Total Amount of Loan
2			$15 000.00
3	1	"= C2* (6.8/100)"	"= C2 + B3"
4	2	"= C2* (6.8/100)"	"= C3 + B4"

I set up a spreadsheet to calculate the interest charged every year and the loan amount. Each year, 6.8% of $15 000, or $1020, will be charged in interest.

	A	B	C
1	Time (Years)	Total Interest Charged	Total Amount of Loan
2			$15 000.00
3	1	$1 020.00	$16 020.00
4	2	$1 020.00	$17 040.00
5	3	$1 020.00	$18 060.00
6	4	$1 020.00	$19 080.00
7	5	$1 020.00	$20 100.00
8	6	$1 020.00	$21 120.00
9	7	$1 020.00	$22 140.00
10	8	$1 020.00	$23 160.00
11	9	$1 020.00	$24 180.00
12	10	$1 020.00	$25 200.00

I used the spreadsheet to calculate the amount Tina would need to pay back at the end of each year during the 10-year period.

Tech Support

For help using a spreadsheet to enter values and formulas, and to fill down, see Technical Appendix, B-21.

EXAMPLE 3 Selecting a strategy to calculate the amount owed after less than a year

Philip borrows $540 for 85 days by taking a cash advance on his credit card. The interest rate is 26%/a simple interest. How much will he need to pay back at the end of the loan period, and how much interest will he have paid?

Lara's Solution

$$t = \frac{85}{365}$$

$$P = \$540$$

$$r = 26\% = 0.26$$

Philip isn't borrowing the money for a full year, so I expressed the time as a fraction of 365 days in a year. I knew the principal, *P*, and I wrote the interest rate, *r*, as a decimal.

$$A = P(1 + rt)$$

$$= (\$540)\left(1 + 0.26 \times \frac{85}{365}\right)$$

To calculate the amount at the end of the loan period, I substituted the values of *P*, *r*, and *t* into the formula.

$$\doteq (\$540)(1.061)$$

$$= \$572.70$$

I rounded to the nearest cent.

Of the $572.70 that Philip has to pay back, $32.70 is interest.

I subtracted the principal to get the interest.

EXAMPLE 4 Calculating the time needed to earn a specific amount on an investment

Tanya invests \$4850 at 7.6%/a simple interest. If she wants the money to increase to \$8000, how long will she need to invest her money?

Josh's Solution

$$P = \$4850$$
$$r = 7.6\% = 0.076$$
$$A = \$8000$$

I knew the principal, the interest rate, and the total amount.

$$A = P + Prt$$
$$8000 = 4850 + (4850)(0.076)t$$
$$8000 = 4850 + 368.6t$$
$$3150 = 368.6t$$

I substituted the values of P, r, and A into the formula and solved for t.

$$t = \frac{3150}{368.6}$$
$$\doteq 8.546$$

When I solved for t, I got a value greater than 8.

$$0.546 \times 365 \text{ days} \doteq 199.2 \text{ days}$$

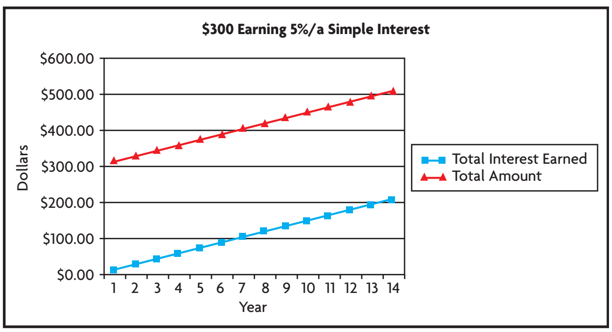
The 8 meant 8 years, so I had to figure out what 0.546 of a year was.

Tanya would have to invest her money for 8 years and 200 days to get \$8000.

In Summary

Key Ideas

- Simple interest is calculated only on the principal.
- The total amount, A, and interest earned, I, are linear functions in terms of time, so their graphs are straight lines (see graph below). The values of A and I at the end of each interest period form the terms of two arithmetic sequences.



(*continued*)

Need to Know

- Simple interest can be calculated using the formula $I = Prt$, where I is the interest; P is the principal; r is the interest rate, expressed as a decimal; and t is time, expressed in the same period as the interest rate, usually per year.
- The total amount, A, of an investment earning simple interest can be calculated using the formula $A = P + Prt$ or $A = P(1 + rt)$.
- Unless otherwise stated, an interest rate is assumed to be per year.

CHECK *Your Understanding*

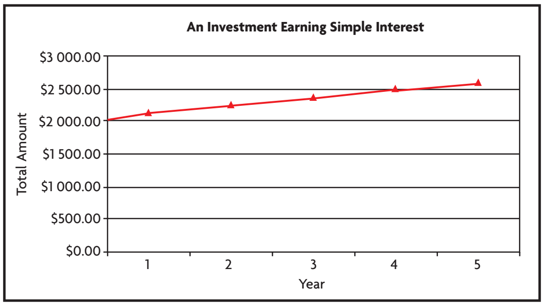
1. Each situation represents an investment earning simple interest. Calculate the total amount at the end of each period.

i) 1st, 2nd, and 3rd years **ii)** 15 years

a) $500 at 6.4%/a
b) $1250 at 4.1%/a
c) $25 000 at 5%/a
d) $1700 at 2.3%/a

2. The graph at the right represents the total amount of an investment of principal $*P* earning a fixed rate of simple interest over a period of 5 years.

a) What is the principal?
b) How much interest is earned in 5 years?
c) What interest rate is being applied?
d) State the equation that represents the amount as a function of time.



3. Michel invests $850 at 7%/a simple interest. How long will he have to leave his investment in the bank before earning $200 in interest?

4. Sally has a balance of $2845 on her credit card. What rate of simple interest is she being charged if she must pay $26.19 interest for the 12 days her payment is late?

PRACTISING

5. For each investment, calculate the interest earned and the total amount.
K

	Principal	Rate of Simple Interest per Year	Time
a)	$500	4.8%	8 years
b)	$3 200	9.8%	12 years
c)	$5 000	3.9%	16 months
d)	$128	18%	5 months
e)	$50 000	24%	17 weeks
f)	$4 500	12%	100 days

6. Mario borrows \$4800 for 8.5 years at a fixed rate of simple interest. At the end of that time, he owes \$8000. What interest rate is he being charged?

7. How much money must be invested at 6.3%/a simple interest to earn \$250 in interest each month?

8. Nina deposits \$3500 into a savings account. The rate of simple interest is 5.5%/a.

a) By how much does the amount in her account increase each year?
b) Determine the amount in her account at the end of each of the first 5 years.
c) State the total amount as the general term of a sequence.
d) Graph this sequence.

9. Ahmad deposits an amount on September 1, 2005, into an account that earns simple interest quarterly. His bank sends him statements after each quarter. The amounts for the first four quarters are shown.

a) How much did Ahmad invest?
b) What rate of simple interest is he earning?

Statement	Date	Balance
1	Dec. 1, 2005	\$3994.32
2	Mar. 1, 2006	\$4248.64
3	Jun. 1, 2006	\$4502.96
4	Sept. 1, 2006	\$4757.28

Year	Amount Owed
1	\$2081.25
2	\$2312.50
3	\$2543.75
4	\$2775.00
5	\$3006.25

10. A Anita borrows some money at a fixed rate of simple interest. The amount she owes at the end of each of the first five years is shown at the left.

a) How much did Anita borrow?
b) State the total amount as the general term of a sequence.
c) How much time will have passed before Anita owes \$7500?

11. T Len invests \$5200 at 3%/a simple interest, while his friend Dave invests \$3600 at 5%/a simple interest. How long will it take for Dave's investment to be worth more than Len's?

12. C Lotti invests some money at a fixed rate of simple interest. She uses the function $A(t) = 750 + 27.75t$ to calculate how much her investment will be worth after t years. How much did she invest and what interest rate is she earning? Explain your reasoning.

Extending

13. The doubling time of an investment is the length of time it takes for the total amount invested to become double the original amount invested. Determine a formula for the doubling time, D, of an investment of principal \$$P$ earning a rate of simple interest of r%/a.

14. Sara's parents decide to invest \$500 on each of her birthdays from the day she is born until she becomes 25. Each investment earns 6.4%/a simple interest. What will be the total amount of the investments when Sara is 25?

8.2 Compound Interest: Future Value

GOAL

Determine the future value of a principal being charged or earning compound interest.

YOU WILL NEED

- graphing calculator
- spreadsheet software (optional)

LEARN ABOUT the Math

Mena invests \$2000 in a bank account that pays 6%/a compounded annually. The savings account is called the "Accumulator" and pays compound interest.

compound interest
interest that is added to the principal *before* new interest earned is calculated. So interest is calculated on the principal *and* on interest already earned. Interest is paid at regular time intervals called the **compounding period.**

compounding period
the intervals at which interest is calculated; for example,
annually ⇒ 1 time per year
semi-annually ⇒ 2 times per year
quarterly ⇒ 4 times per year
monthly ⇒ 12 times per year

? What type of function will model the growth of Mena's money?

A. Calculate the interest earned and amount at the end of the first year. Record your answers in a table as shown.

Year	Balance at Start of Year	Interest Earned	Balance at End of Year
0	—	—	\$2000
1	\$2000		
2			
3			
4			
5			

B. Complete the table for the 2nd to 5th years.

C. Enter your data for Year and Balance at End of Year into either lists on a graphing calculator or columns in a spreadsheet.

D. Create a scatter plot, using Year as the independent variable.

E. What type of function best models the growth of Mena's money? You may need to calculate more data points before you decide. Explain how you know.

F. Determine the function that models the amount of her investment over time.

Reflecting

G. Compare the total amount of Mena's investment with that based on the same principal earning simple interest. What is the advantage of earning compound interest over simple interest?

H. How are compound interest, exponential functions, and geometric sequences related?

APPLY the Math

EXAMPLE 1 Representing any situation earning compound interest as a function

Tim borrows \$5300 at 4.6%/a compounded annually.

a) How much will he have to pay back if he borrows the money for 10 years?

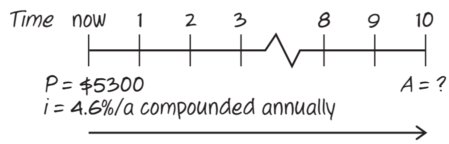
b) Determine the future value, A, and interest earned, I, if he invested a principal of \$$P$ for n years at i%/a compounded annually.

future value

the total amount, A, of an investment after a certain length of time

Shelley's Solution

a)



I drew a timeline for the situation.

end of 1st year:

$$A = P(1 + rt)$$
$$= 5300[1 + 0.046(1)]$$
$$= 5300(1.046)$$
$$= \$5543.80$$

At the end of the first year, Tim gets charged 4.6% of his original \$5300 loan. So I calculated the amount he would owe at the end of that year.

end of 2nd year:

$$A = 5543.80[1 + 0.046(1)]$$
$$= 5543.80(1.046)$$
$$\doteq \$5798.81$$

At the end of the second year, he gets charged 4.6% of the amount he owed at the end of the first year, \$5543.80. So I calculated the amount he would owe at the end of that year.

end of 3rd year:

$$A = 5798.81[1 + 0.046(1)]$$
$$= 5798.81(1.046)$$
$$= \$6065.56$$

I used the same method to calculate the amount at the end of the third year. Each time, I rounded to the nearest cent.

$$t_1 = 5300 \times 1.046^1 = \$5543.80$$
$$t_2 = 5300 \times 1.046^2 \doteq \$5798.81$$
$$t_3 = 5300 \times 1.046^3 \doteq \$6065.56$$
$$\vdots$$
$$t_n = 5300 \times 1.046^n$$

I noticed that I was multiplying by 1.046 each time. This is a geometric sequence with common ratio 1.046. The general term of the sequence is $t_n = 5300 \times 1.046^n$. I used this formula to calculate the first three terms, and I got the same numbers as in my previous calculations.

$$t_{10} = 5300 \times 1.046^{10}$$
$$\doteq \$8309.84$$

To determine how much Tim would owe after 10 years, I substituted $n = 10$ into the formula for the general term.

Tim would have to pay back \$8309.84 after 10 years.

b) end of 1st year:

$$A = P(1 + in)$$
$$A_1 = P[1 + i(1)]$$
$$= P(1 + i)$$

end of 2nd year:

$$A_2 = [P(1 + i)](1 + i)$$
$$= P(1 + i)^2$$

end of 3rd year:

$$A_3 = [P(1 + i)^2](1 + i)$$
$$= P(1 + i)^3$$

end of nth year:

$$A = P(1 + i)^n$$

For compound interest, the amount or future value depends on time, as it did for simple interest. Since the interest rate is i%/a, I substituted i for r into the formula for the total amount and calculated the future value for the 1st, 2nd, and 3rd years.

This is a geometric sequence with first term $P(1 + i)$ and common ratio $1 + i$, so I wrote the general term, which gave me the amount after n years. The amount is an exponential function in terms of time.

$$I = A - P$$
$$= P(1 + i)^n - P$$
$$= P[(1 + i)^n - 1]$$

To determine the interest earned over a period of n years, I subtracted the principal from the total amount.

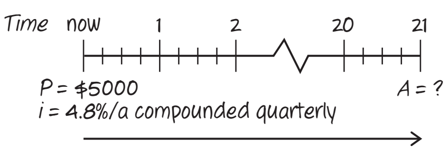
I factored out the common factor P.

The future value of an investment of $\$P$ for n years at i%/a compounded annually will be $A = P(1 + i)^n$, and the total interest earned will be $I = P[(1 + i)^n - 1]$.

EXAMPLE 2 — Selecting a strategy to determine the amount when the compounding period is less than a year

Lara's grandparents invested $5000 at 4.8%/a compounded quarterly when she was born. How much will the investment be worth on her 21st birthday?

Herman's Solution



I drew a timeline for the situation.

$P = \$5000$

$i = 0.048 \div 4$

$= 0.012$

Since interest is paid quarterly for each compounding period, I divided the annual interest rate by 4 to get the interest rate.

$n = 21 \times 4$

$= 84$

Interest is paid 4 times per year, so I calculated the number of compounding periods. This is the total number of times that the interest would be calculated over the 21 years.

$A = P(1 + i)^n$

I used the formula $A = P(1 + i)^n$, where i is the interest rate per compounding period and n is the number of compounding periods.

$A = 5000(1 + 0.012)^{84}$

$\doteq \$13\,618.62$

I noticed that solving this problem is the same as solving a problem in which the money earns 1.2%/a compounded annually for 84 years.

The \$5000 investment will be worth \$13 618.62 on Lara's 21st birthday.

EXAMPLE 3 Calculating the difference of the amounts of two different investments

On her 15th birthday, Trudy invests \$10 000 at 8%/a compounded monthly. When Lina turns 45, she invests \$10 000 at 8%/a compounded monthly. If both women leave their investments until they are 65, how much more will Trudy's investment be worth?

Henry's Solution

$P = \$10\,000$

$i = \frac{0.08}{12}$

To calculate how much Trudy's investment will be worth when she turns 65, I first determined the interest rate per month as a fraction.

$n = (65 - 15) \times 12$

$= 600$

Since interest is compounded monthly and she is investing for 50 years, there will be $50 \times 12 = 600$ compounding periods.

$A = 10\,000\left(1 + \frac{0.08}{12}\right)^{600}$

$\doteq \$538\,781.94$

Trudy's investment will be worth \$538 781.94 when she turns 65.

$P = \$10\ 000$

$i = \frac{0.08}{12}$

Lina's investment has the same principal and interest rate per month as Trudy's, but fewer compounding periods.

$n = (65 - 45) \times 12$

$= 240$

Since Lina invested for 20 years, there will be $20 \times 12 = 240$ compounding periods.

$A = 10\ 000\left(1 + \frac{0.08}{12}\right)^{240}$

$\doteq \$49\ 268.03$

Lina's investment will be worth \$49 268.03 when she turns 65.

$\$538\ 781.94 - \$49\ 268.03 = \$489\ 513.91$

I subtracted Lina's amount from Trudy's amount.

Trudy's investment will be worth \$489 513.91 more than Lina's.

EXAMPLE 4 Comparing simple interest and compound interest

Nicolas invests \$1000. How long would it take for his investment to double for each type of interest earned?

a) 5%/a simple interest

b) 5%/a compounded semi-annually

Jesse's Solution

a) $P = \$1000$

$r = 5\% = 0.05$

I knew the principal and the interest rate.

$I = \$1000$

$I = Prt$

Since Nicolas's investment will double and he is earning simple interest, the interest earned must be the same as the principal.

$1000 = 1000(0.05)t$

I substituted the values of *P*, *r*, and *I* into the formula for the interest earned.

$\frac{1000}{1000(0.05)} = \left(\frac{1000(0.05)}{1000(0.05)}\right)t$

To solve for *t*, I divided both sides of the equation by 1000(0.05).

$20 = t$

It will take 20 years for Nicolas's investment to double at 5%/a simple interest.

b) $i = \frac{0.05}{2} = 0.025$

$A = P(1 + i)^n$

Since interest is paid semi-annually, I divided the annual interest rate by 2 to get the interest rate per half year.

$A = 1000(1 + 0.025)^{40}$

$\doteq \$2685.06$

I substituted $P = 1000$ and $i = 0.025$ into the formula for the amount. Then I used guess-and-check to determine n. I tried 20 years, or $n = 20 \times 2 = 40$ compounding periods.

$A = 1000(1 + 0.025)^{20}$

$\doteq \$1638.62$

The amount after 20 years was too much.

Next I tried 10 years, or $n = 10 \times 2 = 20$ compounding periods, but that wasn't enough.

$A = 1000(1 + 0.025)^{28}$

$\doteq \$1996.50$

Since my second guess was slightly closer to \$2000 than my first guess, I tried 14 years, or $n = 14 \times 2 = 28$ compounding periods. The result was close to double.

	A	B	C
1	Year	Simple Interest	Compound Interest
2	0	$1 000.00	$1 000.00
3	0.5	"=B2 + 1000*0.05/2"	"=C2*(1 + 0.025)"
4	1	"=B3 + 1000*0.05/2"	"=C3*(1 + 0.025)"

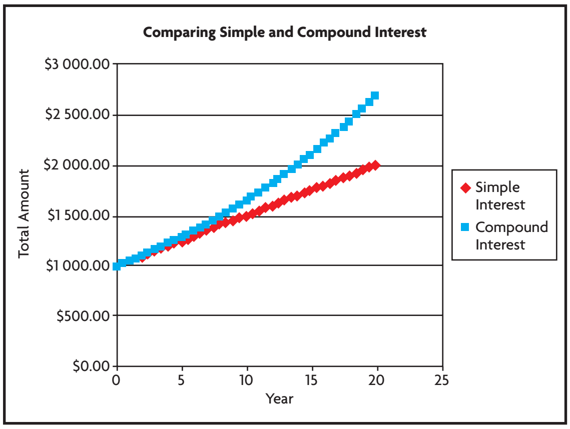
I then used a spreadsheet to check my result.

	A	B	C
1	Year	Simple Interest	Compound Interest
2	0	$1 000.00	$1 000.00
3	0.5	$1 025.00	$1 025.00
4	1	$1 050.00	$1 050.63
5	1.5	$1 075.00	$1 076.89
6	2	$1 100.00	$1 103.81
7	2.5	$1 125.00	$1 131.41
8	3	$1 150.00	$1 159.69

28	13	$1 650.00	$1 900.29
29	13.5	$1 675.00	$1 947.80
30	14	$1 700.00	$1 996.50
31	14.5	$1 725.00	$2 046.41

40	19	$1 950.00	$2 555.68
41	19.5	$1 975.00	$2 619.57
42	20	$2 000.00	$2 685.06

The investment earning simple interest took 20 years to double. If Nicolas earns compound interest, he gets interest on interest previously earned, so his investment grows faster. I used the spreadsheet to compare the two possibilities. With compound interest, the investment almost doubles after 14 years.



I graphed the amount of the investment for both cases. From the graph, the simple-interest situation is modelled by a linear function growing at a constant rate, while the compound-interest situation is modelled by an exponential function growing at a changing rate.

It will take about 14 years for Nicolas's investment to double at 5%/a compounded semi-annually.

Tech Support

For help using a spreadsheet to graph functions, see Technical Appendix, B-21.

In Summary

Key Ideas

- Compound interest is calculated by applying the interest rate to the principal and any interest already earned.
- The total amounts at the end of each interest period form a geometric sequence. So compound interest results in exponential growth.
- The total amount, A, of an investment after a certain period is called the future value of the investment.

Need to Know

- Banks pay or charge compound interest at regular intervals called the compounding period. If interest is compounded annually, then at the end of the first year, interest is calculated and added to the principal. At the end of the second year, interest is calculated on the new balance (principal plus interest earned from the previous year). This pattern continues every year the investment is kept.
- The future value of an investment earning compound interest can be calculated using the formula $A = P(1 + i)^n$, where A is the future value; P is the principal; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.
- The most common compounding periods are:

annually	1 time per year	i = annual interest rate	n = number of years
semi-annually	2 times per year	i = annual interest rate ÷ 2	n = number of years × 2
quarterly	4 times per year	i = annual interest rate ÷ 4	n = number of years × 4
monthly	12 times per year	i = annual interest rate ÷ 12	n = number of years × 12

- Compound interest can be calculated using the formula $I = A - P$ or $I = P[(1 + i)^n - 1]$, where I is the total interest.

CHECK Your Understanding

1. Copy and complete the table.

	Rate of Compound Interest per Year	Compounding Period	Time	Interest Rate per Compounding Period, *i*	Number of Compounding Periods, *n*
a)	5.4%	semi-annually	5 years		
b)	3.6%	monthly	3 years		
c)	2.9%	quarterly	7 years		
d)	2.6%	weekly	10 months		

2. **i)** Determine the amount owed at the end of each of the first five compounding periods.

ii) Determine the general term for the amount owed at the end of the *n*th compounding period.

	Amount Borrowed	Rate of Compound Interest per Year	Compounding Period
a)	$10 000	7.2%	annually
b)	$10 000	3.8%	semi-annually
c)	$10 000	6.8%	quarterly
d)	$10 000	10.8%	monthly

3. Calculate the future value of each investment. Draw a timeline for each.

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	$258	3.5%	annually	10 years
b)	$5 000	6.4%	semi-annually	20 years
c)	$1 200	2.8%	quarterly	6 years
d)	$45 000	6%	monthly	25 years

PRACTISING

4. For each investment, determine the future value and the total interest earned.
K

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	$4 000	3%	annually	4 years
b)	$7 500	6%	monthly	6 years
c)	$15 000	2.4%	quarterly	5 years
d)	$28 200	5.5%	semi-annually	10 years
e)	$850	3.65%	daily	1 year
f)	$2 225	5.2%	weekly	47 weeks

Year	Total Amount
1	$4240.00
2	$4494.40
3	$4764.06

5. Sima invests some money in an account that earns a fixed rate of interest compounded annually. The amounts of the investment at the end of the first three years are shown at the right.
 a) Determine the annual rate of compound interest earned.
 b) How much did Sima invest?

6. Chris invests $10 000 at 7.2%/a compounded monthly. How long will it take for his investment to grow to $25 000?

7. **A** Serena wants to borrow $15 000 and pay it back in 10 years. Interest rates are high, so the bank makes her two offers:
- Option 1: Borrow the money at 10%/a compounded quarterly for the full term.
- Option 2: Borrow the money at 12%/a compounded quarterly for 5 years and then renegotiate the loan based on the new balance for the last 5 years.

If, in 5 years, the interest rate will be 6%/a compounded quarterly, how much will Serena save by choosing the second option?

8. Ted used the exponential function $A(n) = 5000 \times 1.0075^{12n}$ to represent the future value, A, in dollars, of an investment. Determine the principal, the annual interest rate, and the compounding period. Explain your reasoning.

9. Margaret can finance the purchase of a $949.99 refrigerator one of two ways:
- Plan A: 10%/a simple interest for 2 years
- Plan B: 5%/a compounded quarterly for 2 years

Which plan should she choose? Justify your answer.

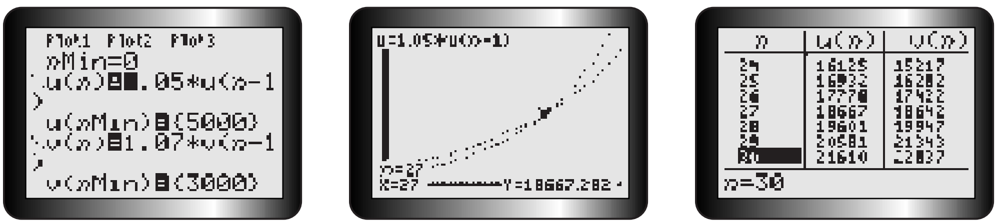
10. Eric bought a $1000 Canada Savings Bond that earns 5%/a compounded annually. Eric can redeem the bond in 7 years. Determine the future value of the bond.

11. Dieter deposits $9000 into an account that pays 10%/a compounded quarterly. After three years, the interest rate changes to 9%/a compounded semi-annually. Calculate the value of his investment two years after this change.

12. Cliff has some money he wants to invest for his retirement. He is offered two options:
- 10%/a simple interest
- 5%/a compounded annually

Under what conditions should he choose the first option?

13. **T** Noreen used her graphing calculator to investigate two sequences. Three screenshots from her investigation are shown. Create a problem for this situation and solve it.



14. You are searching different banks for the best interest rate on an investment, and you find these rates:
- 6.6%/a compounded annually
- 6.55%/a compounded semi-annually
- 6.5%/a compounded quarterly
- 6.45/a compounded monthly

Rank the rates from most to least return on your investment.

15. On July 1, 1996, Anna invested $2000 in an account that earned 6%/a compounded monthly. On July 1, 2001, she moved the total amount to a new account that paid 8%/a compounded quarterly. Determine the balance in her account on January 1, 2008.

16. Bernie deposited $4000 into an account that pays 4%/a compounded quarterly during the first year. The interest rate on this account is then increased by 0.2% each year. Calculate the balance in Bernie's account after three years.

17. On the day Rachel was born, her grandparents deposited $500 into a savings account that earns 4.8%/a compounded monthly. They deposited the same amount on her 5th, 10th, and 15th birthdays. Determine the balance in the account on Rachel's 18th birthday.

18. **C** Create a mind map for the concept of *interest*. Show how the calculations of simple and compound interest are related to functions and sequences.

Extending

19. Liz decides to save money to buy an electric car. She invests $500 every 6 months at 6.8%/a compounded semi-annually. What total amount of money will she have at the end of the 10th year?

20. An effective annual interest rate is the interest rate that is equivalent to the given one, assuming that compounding occurs annually. Calculate the effective annual interest rate for each loan. Round to two decimal places.

	Rate of Compound Interest per Year	Compounding Period
a)	6.3%	semi-annually
b)	4.2%	monthly
c)	3.2%	quarterly

8.3 Compound Interest: Present Value

GOAL

Determine the present value of an amount being charged or earning compound interest.

YOU WILL NEED

- graphing calculator
- spreadsheet software

LEARN ABOUT the Math

Anton's parents would like to put some money away so that he will have \$15 000 to study music professionally in 10 years. They can earn 6%/a compounded annually on their investment.

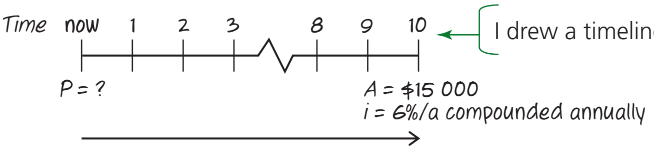
? How much money should Anton's parents invest now so that it will grow to \$15 000 in 10 years at 6%/a compounded annually?

EXAMPLE 1 Selecting a strategy to determine the principal for a given amount

Determine the present value of Anton's parents' investment if it must be worth \$15 000 ten years from now.

present value
the principal that would have to be invested now to get a specific future value in a certain amount of time. *PV* is used for present value instead of *P*, since *P* is used for principal.

Tina's Solution: Working Backward



I drew a timeline for the situation.

end of 1st year:

$I = 0.06P$

$A = P + 0.06P$

$= 1.06P$

The interest rate is 6%/a compounded annually. I calculated the interest earned and the amount at the end of the first year.

end of 2nd year:

$I = 0.06(1.06P)$

$A = 1.06P + 0.06(1.06P)$

$= 1.06P(1 + 0.06)$

$= 1.06P(1.06)$

$= 1.06^2P$

For the second year, interest is earned on the amount at the end of the first year. I calculated the interest earned and the amount at the end of the second year.

$1.06P, 1.06^2P, 1.06^3P, \ldots, 1.06^{10}P$

Since 6% is added at the end of each year, I got 106% of what I started with. So multiplying by 1.06 gives the next term of the sequence. The amounts at the end of each year form a geometric sequence with common ratio 1.06. The amount at the end of the 10th year has an exponent of 10.

end of 10th year:

$$15\,000 = 1.06^{10}P$$

Since Anton's parents want \$15 000 at the end of 10 years, I set the 10th term of the sequence equal to \$15 000 and solved for *P*.

$$P = \frac{15\,000}{1.06^{10}}$$

$$\doteq \$8375.92$$

Anton's parents would need to invest \$8375.92 now to get \$15 000 in 10 years.

Jamie's Solution: Representing the Formula for the Amount in a Different Way

$$A = \$15\,000$$

$$i = 6\% = 0.06$$

$$n = 10$$

Anton's parents invest a certain amount and let it grow to \$15 000 at 6%/a compounded annually.

$$A = PV(1 + i)^n$$

An investment earning compound interest grows like an exponential function, so I wrote the amount as an exponential formula.

$$PV = \frac{A}{(1 + i)^n}$$

To calculate the principal that Anton's parents would have to invest, I rearranged the formula, substituted, and solved for *PV*.

$$= \frac{15\,000}{(1 + 0.06)^{10}}$$

$$\doteq \$8375.92$$

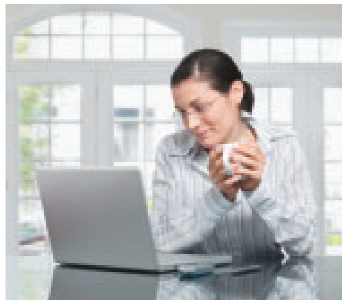
Anton's parents would need to invest \$8375.92 now to get \$15 000 in 10 years.

Reflecting

A. How are the problems of determining the present value and the amount of an investment related?

B. Based on Example 1, which method do you prefer to use to calculate the present value? Why?

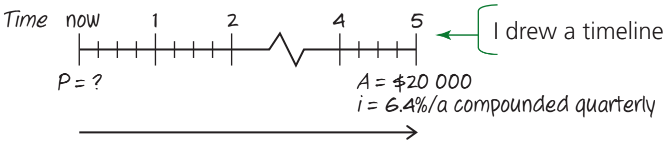
APPLY the Math



EXAMPLE 2 Solving a problem involving present value

Monica wants to start a business and needs to borrow some money. Her bank will charge her 6.4%/a compounded quarterly. Monica wants to repay the loan in 5 years, but doesn't want the amount she pays back to be more than \$20 000. What is the maximum amount that she can borrow and how much interest will she pay if she doesn't pay anything back until the end of the 5 years?

Kwok's Solution



I drew a timeline for the situation.

$$i = \frac{0.064}{4} = 0.016$$

$$n = 5 \times 4 = 20$$

I calculated the interest rate Monica would be charged each compounding period and how many periods the loan would last.

$$PV = \frac{A}{(1 + i)^n}$$

$$= \frac{20\ 000}{(1 + 0.016)^{20}}$$

$$\doteq \$14\ 559.81$$

Next I calculated the present value of the \$20 000 at the given interest rate.

$$I = A - PV$$

$$= \$20\ 000 - \$14\ 559.81$$

$$= \$5440.19$$

I determined the interest by subtracting the present value from the amount.

The most Monica can borrow is \$14 559.81; she will pay \$5440.19 in interest.

EXAMPLE 3 Selecting a strategy to determine the interest rate

Tony is investing \$5000 that he would like to grow to at least \$50 000 by the time he retires in 40 years. What annual interest rate, compounded annually, will provide this? Round your answer to two decimal places.

Philip's Solution: Using a Graphing Calculator

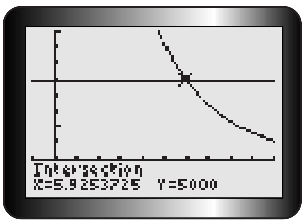
$$PV = \$5000$$
$$n = 40$$
$$A = 50\ 000$$

I knew the principal, the amount (or future value), and the number of years.

$$PV = \frac{A}{(1 + i)^n}$$
$$5000 = \frac{50\ 000}{(1 + i)^{40}}$$

I wrote the formula for the present value and substituted the given information. To calculate i, I thought of the intersection of two functions: $Y1 = \frac{50\ 000}{(1 + \frac{X}{100})^{40}}$ and $Y2 = 5000$.

I entered these into my graphing calculator. I used Y1 and Y2 for the present value and X for the interest rate.



I graphed both functions on the same graph and used the calculator to find the point of intersection.

Tony would need to get at least 5.93%/a compounded annually to reach his goal.

Tech Support

For help using a graphing calculator to find the point of intersection of two functions, see Technical Appendix, B-12.

Derek's Solution: Using the Formula

$$PV = \$5000$$
$$n = 40$$
$$A = \$50\ 000$$

I knew the principal, the amount (or future value), and the number of years.

$$PV = \frac{A}{(1 + i)^n}$$
$$5000 = \frac{50\ 000}{(1 + i)^{40}}$$

I wrote the formula for the present value, then substituted the given information, and rearranged to solve for *i*.

$$5000(1 + i)^{40} = 50\ 000$$
$$(1 + i)^{40} = 10$$

I multiplied both sides of the equation by $(1 + i)^{40}$, then I divided both sides by 5000.

$$1 + i = \sqrt[40]{10}$$
$$1 + i \doteq 1.0593$$
$$i = 0.0593$$
$$i = 5.93\%$$

To calculate *i*, I used the inverse operation of raising something to the 40th power, which is determining the 40th root.

Tony would need to get at least 5.93%/a compounded annually to reach his goal.

In Summary

Key Idea

- The principal, *PV*, that must be invested now to grow to a specific future value, *A*, is called the present value.

Need to Know

- The present value of an investment earning compound interest can be calculated using the formula $PV = \frac{A}{(1 + i)^n}$ or $PV = A(1 + i)^{-n}$, where *PV* is the present value; *A* is the total amount, or future value; *i* is the interest rate per compounding period, expressed as a decimal; and *n* is the number of compounding periods.

CHECK Your Understanding

1. Calculate the present value of each investment.

	Rate of Compound Interest per Year	Compounding Period	Time	Future Value
a)	4%	annually	10 years	$10 000
b)	6.2%	semi-annually	5 years	$100 000
c)	5.2%	quarterly	15 years	$23 000
d)	6.6%	monthly	100 years	$2 500

2. Kevin and Lui both want to have $10 000 in 20 years. Kevin can invest at 5%/a compounded annually and Lui can invest at 4.8%/a compounded monthly. Who has to invest more money to reach his goal?

PRACTISING

3. For each investment, determine the present value and the interest earned.
K

	Rate of Compound Interest per Year	Compounding Period	Time	Future Value
a)	6%	annually	4 years	$10 000
b)	8.2%	semi-annually	3 years	$6 200
c)	5.6%	quarterly	15 years	$20 000
d)	4.2%	monthly	9 years	$12 800

4. Chandra borrows some money at 7.2%/a compounded annually. After 5 years, she repays $12 033.52 for the principal and the interest. How much money did Chandra borrow?

5. Nazir saved $900 to buy a plasma TV. He borrowed the rest at an interest rate of 18%/a compounded monthly. Two years later, he paid $1429.50 for the principal and the interest. How much did the TV originally cost?

6. Rico can invest money at 10%/a compounded quarterly. He would like $15 000 in 10 years. How much does he need to invest now?

7. Colin borrowed some money at 7.16%/a compounded quarterly. Three years later, he paid $5000 toward the principal and the interest. After another two years, he paid another $5000. After another five years, he paid the remainder of the principal and the interest, which totalled $5000. How much money did he originally borrow?

8. Tia is investing $2500 that she would like to grow to $6000 in 10 years. At what annual interest rate, compounded quarterly, must Tia invest her money? Round your answer to two decimal places.

9. **A** Franco invests some money at 6.9%/a compounded annually and David invests some money at 6.9%/a compounded monthly. After 30 years, each investment is worth $25 000. Who made the greater original investment and by how much?

10. **T** Sally invests some money at 6%/a compounded annually. After 5 years, she takes the principal and interest and reinvests it all at 7.2%/a compounded quarterly for 6 more years. At the end of this time, her investment is worth $14 784.56. How much did Sally originally invest?

11. Steve wants to have $25 000 in 25 years. He can get only 3.2%/a interest compounded quarterly. His bank will guarantee the rate for either 5 years or 8 years.
    - In 5 years, he will probably get 4%/a compounded quarterly for the remainder of the term.
    - In 8 years, he will probably get 5%/a compounded quarterly for the remainder of the term.

    a) Which guarantee should Steve choose, the 5-year one or the 8-year one?
    b) How much does he need to invest?

12. **C** Describe how determining the present value of an investment is similar to solving a radioactive decay problem.

Extending

13. Louise invests $5000 at 5.4%/a compounded semi-annually. She would like the money to grow to $12 000. How long will she have to wait?

14. What annual interest rate, compounded quarterly, would cause an investment to triple every 10 years? Round your answer to two decimal places.

15. You buy a home entertainment system on credit. You make monthly payments of $268.17 for $2\frac{1}{2}$ years and are charged 19.2%/a interest compounded monthly. How much interest will you have paid on your purchase?

16. Determine a formula for the present value of an investment with future value, A, earning simple interest at a rate of i% per interest period for n interest periods.

Curious Math

The Rule of 72

Working with compound interest can be difficult if you don't have a calculator handy. For years, banks, investors, and the general public have used "the rule of 72" to help approximate calculations involving compound interest. Here is how the rule works:

If an investment is earning r%/a compounded annually, then it will take $72 \div r$ years to double in value.

So if you are earning 8%/a compounded annually, the rule indicates that it will take $72 \div 8 = 9$ years for the money to double in value. Suppose you invested $1000. Then the formula for the future value gives

$$\begin{aligned} A &= P(1 + i)^n \\ &= 1000(1.08)^9 \\ &\doteq \$1999.00 \end{aligned}$$

which is very close to double your money. The spreadsheet below shows how the time, in years, predicted by the rule of 72 is very close to the actual doubling time.

	A	B	C
1	Interest Rate	Double Time Using the Rule of 72	Actual Double Time
2	1	72.00	69.66
3	2	36.00	35.00
4	3	24.00	23.45
5	4	18.00	17.67
6	5	14.40	14.21
7	6	12.00	11.90
8	7	10.29	10.24
9	8	9.00	9.01
10	9	8.00	8.04
11	10	7.20	7.27
12	11	6.55	6.64
13	12	6.00	6.12

1. How could you use the rule of 72 to determine how much a $1000 investment earning 8%/a compounded annually would be worth after 45 years?
2. How close is your estimate to the actual amount after 45 years?

8 Mid-Chapter Review

FREQUENTLY ASKED Questions

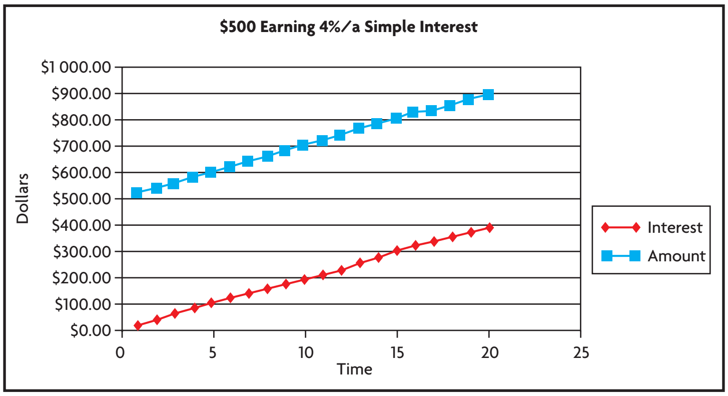
Q: What strategies can you use to solve problems involving simple interest?

A1: Since simple interest is paid only on the principal, both the interest, I, and the amount, A, grow at a constant rate. So I and A can be modelled by linear functions in terms of time or arithmetic sequences.

EXAMPLE

If \$500 is invested at 4%/a simple interest, then 4% of \$500, or \$20, interest is earned each year. You can model the interest and the amount by the functions $I(t) = 20t$ and $A(t) = 500 + 20t$, where I and A are in dollars and t is time in years.

	A	B	C
1	Year	Interest	Amount
2	1	\$20.00	\$520.00
3	2	\$40.00	\$540.00
4	3	\$60.00	\$560.00
5	4	\$80.00	\$580.00
6	5	\$100.00	\$600.00
7	6	\$120.00	\$620.00
8	7	\$140.00	\$640.00
9	8	\$160.00	\$660.00
10	9	\$180.00	\$680.00
11	10	\$200.00	\$700.00
12	11	\$220.00	\$720.00
13	12	\$240.00	\$740.00
14	13	\$260.00	\$760.00
15	14	\$280.00	\$780.00
16	15	\$300.00	\$800.00
17	16	\$320.00	\$820.00
18	17	\$340.00	\$840.00
19	18	\$360.00	\$860.00
20	19	\$380.00	\$880.00
21	20	\$400.00	\$900.00



A2: To calculate the interest, use the formula $I = Prt$, where I is the interest; P is the principal; r is the interest rate, expressed as a decimal; and t is time, expressed in the same period as the interest rate. To calculate the amount, use the formula $A = P + Prt$ or $A = P(1 + rt)$, where A is the total amount. The formulas for I and A are linear in terms of time.

Study *Aid*

- See Lesson 8.1, Examples 1 to 4.
- Try Mid-Chapter Review Questions 1 to 3.

Q: How can you solve problems involving compound interest?

A1: Compound interest is paid at regular intervals, called compounding periods, and is added to the principal. So interest is calculated on the principal *and* interest already earned. After each compounding period, the total amount, A, or future value, grows by a fixed rate. So A can be modelled by an exponential function in terms of time or a geometric sequence.

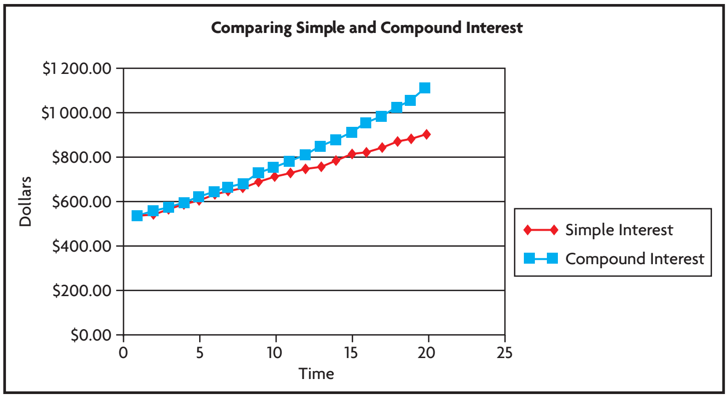
Study *Aid*

- See Lesson 8.2, Examples 1 to 4.
- Try Mid-Chapter Review Questions 4 to 6.

EXAMPLE

If \$500 is invested at 4%/a compounded annually, then 4% of the total amount is earned as interest each year. So the amount can be modelled by $A(t) = 500 \times 1.04^t$, where A is the future value in dollars and t is time in years. The amount grows exponentially, faster than the linear growth of simple interest.

	A	B	C
1	Year	Simple Interest	Compound Interest
2	1	\$520.00	\$520.00
3	2	\$540.00	\$540.80
4	3	\$560.00	\$562.43
5	4	\$580.00	\$584.93
6	5	\$600.00	\$608.33
7	6	\$620.00	\$632.66
8	7	\$640.00	\$657.97
9	8	\$660.00	\$684.28
10	9	\$680.00	\$711.66
11	10	\$700.00	\$740.12
12	11	\$720.00	\$769.73
13	12	\$740.00	\$800.52
14	13	\$760.00	\$832.54
15	14	\$780.00	\$865.84
16	15	\$800.00	\$900.47
17	16	\$820.00	\$936.49
18	17	\$840.00	\$973.95
19	18	\$860.00	\$1012.91
20	19	\$880.00	\$1 053.42
21	20	\$900.00	\$1 095.56



A2: To calculate the total amount, or future value, use the formula $A = P(1 + i)^n$, where A is the future value; P is the principal; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods. This formula is exponential in terms of the number of compounding periods.

Study *Aid*

- See Lesson 8.3, Examples 1 to 3.
- Try Mid-Chapter Review Questions 7 to 9.

Q: For problems involving compound interest, what is the difference between present value and future value?

A: Present value, PV, is the amount you start with (the principal), while future value, A, is the amount you end up with after the last compounding period. These two values are related through the formulas

$$A = P(1 + i)^n \quad \text{and}$$

$$PV = \frac{A}{(1 + i)^n} \quad \text{or} \quad PV = A(1 + i)^{-n}$$

which are just rearrangements of each other. When you solve for PV, you are determining the principal that should be invested to yield the desired amount.

PRACTICE Questions

Lesson 8.1

1. Each situation represents a loan being charged simple interest. Calculate the interest being charged and the total amount.

	Principal	Rate of Simple Interest per Year	Time
a)	$5 400	6.7%	15 years
b)	$400	9.6%	16 months
c)	$15 000	14.3%	80 weeks
d)	$2 500	27.1%	150 days

2. How long would you have to invest $5300 at 7.2%/a simple interest to earn $1200 interest?

3. Tom borrows some money and is charged simple interest on the principal. The balances from his statements for the first three months are shown.

Statement	Balance
1	$1014.60
2	$1079.20
3	$1143.80

a) How much interest is he being charged each month?
b) How much did Tom borrow?
c) What interest rate is he being charged?

Lesson 8.2

4. For each investment, calculate the future value and the total interest earned.

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	$6 300	4.9%	annually	7 years
b)	$14 000	8.8%	semi-annually	10.5 years
c)	$120 000	4.4%	quarterly	44 years
d)	$298	22.8%	monthly	1.5 years

5. George invests $15 000 at 7.2%/a compounded monthly. How long will it take for his investment to grow to $34 000?

6. Sara buys a washer and dryer for $2112. She pays $500 and borrows the remaining amount. A year and a half later, she pays off the loan, which amounted to $2112. What annual interest rate, compounded semi-annually, was Sara being charged? Round your answer to two decimal places.

Lesson 8.3

7. How much money does Maria need to invest at 9.2%/a compounded quarterly in order to have $25 000 after 25 years?

8. Clive inherits an investment that his grandparents made at 7.4%/a compounded semi-annually. The investment was worth $39 382.78 when they took it out 65 years ago. How much did Clive's grandparents invest?

9. Iris borrowed some money at a fixed rate of compound interest, but she forgot what the interest rate was. She knew that the interest was compounded semi-annually. The balances of her first two statements are shown.

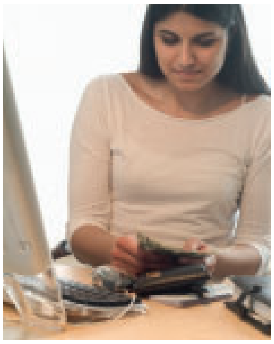
Statement	Time	Balance
1	6 months	$8715.91
2	1 year	$9125.56

a) What interest rate is she being charged? Round your answer to two decimal places.
b) How much did Iris borrow?

8.4 Annuities: Future Value

YOU WILL NEED

- graphing calculator
- spreadsheet software



GOAL

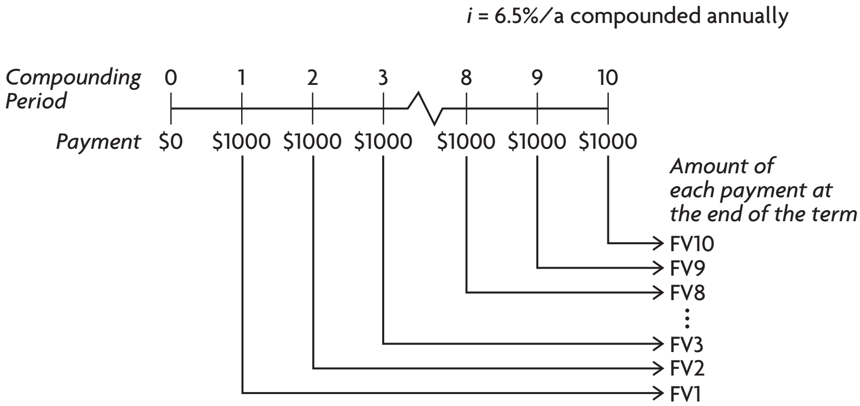
Determine the future value of an annuity earning compound interest.

INVESTIGATE the Math

Christine decides to invest $1000 at the end of each year in a Canada Savings Bond earning 6.5%/a compounded annually. Her first deposit is on December 31, 2007.

? How much will her investments be worth 10 years later, on January 1, 2017?

A. Copy the timeline shown. How would you calculate each of the future values FV1 to FV10?



B. Set up a spreadsheet with columns as shown. Copy the data already entered. Complete the entries under Date Invested up to Dec. 31, 2007.

	A	B	C	D
1	Date Invested	Amount Invested	Number of Years Invested	Value on Jan. 1, 2017
2	Dec. 31, 2016	$1 000.00	0	$1 000.00
3	Dec. 31, 2015	$1 000.00	1	
4	Dec. 31, 2014	$1 000.00	2	

C. Fill in cells D3 and D4 to show what the investments made on Dec. 31, 2015, and Dec. 31, 2014, respectively will be worth on Jan. 1, 2017.

D. How is the value in cell D3 (FV9) related to the value in cell D2 (FV10)? How is the value in cell D4 (FV8) related to the value in cell D3 (FV9)?

E. Use the pattern from part D to complete the rest of the entries under Value on Jan. 1, 2017.

F. What type of sequence do the values on Jan. 1, 2017 form?

G. Calculate the total amount of all the investments at the end of 10 years for this annuity.

annuity
a series of payments or investments made at regular intervals. A **simple** annuity is an annuity in which the payments coincide with the compounding period, or *conversion* period. An **ordinary** annuity is an annuity in which the payments are made at the end of each interval. Unless otherwise stated, each annuity in this chapter is a simple, ordinary annuity.

Reflecting

H. The values of all of the investments at the end of each year for 10 years formed a specific type of sequence. How is the total value of the annuity at the end of 10 years related to a series?

I. How could you use the related series to solve problems involving annuities?

APPLY the Math

EXAMPLE 1 Representing the future value of an annuity earning compound interest as a series

a) Hans plans to invest \$1000 at the end of each 6-month period in an annuity that earns 4.8%/a compounded semi-annually for the next 20 years. What will be the future value of his annuity?

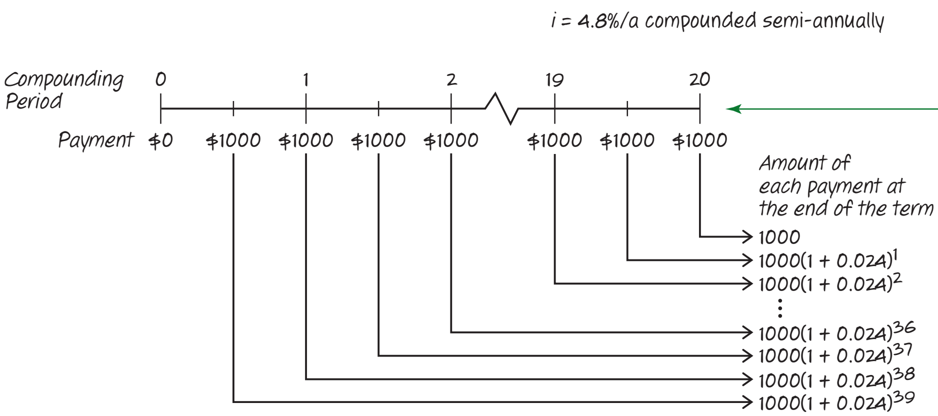
b) You plan to invest \$$R$ at regular intervals in an annuity that earns i% compounded at the end of each interval. What will be the future value, FV, of your annuity after n intervals?

Barbara's Solution

a) $i = \frac{0.048}{2} = 0.024$

$n = 20 \times 2 = 40$

Since the interest is paid semi-annually, I calculated the interest rate per compounding period and the number of compounding periods.



I drew a timeline of the investments for each compounding period, and I represented the amount of each investment.

The last \$1000 investment earned no interest because it was deposited at the end of the term.

The first \$1000 investment earned interest over 39 periods. It didn't earn interest during the first compounding period because it was deposited at the end of that period.

$$1000,\ 1000(1.024),\ 1000(1.024)^2,\ \dots,\ 1000(1.024)^{38},\ 1000(1.024)^{39}$$

The future values of all of the investments form a geometric sequence with first term \$1000 and common ratio 1.024.

$$S_{40} = 1000 + 1000(1.024) + 1000(1.024)^2 + \dots + 1000(1.024)^{38} + 1000(1.024)^{39}$$

The total amount of all these investments is the first 40 terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{40} = \frac{1000(1.024^{40} - 1)}{1.024 - 1}$$

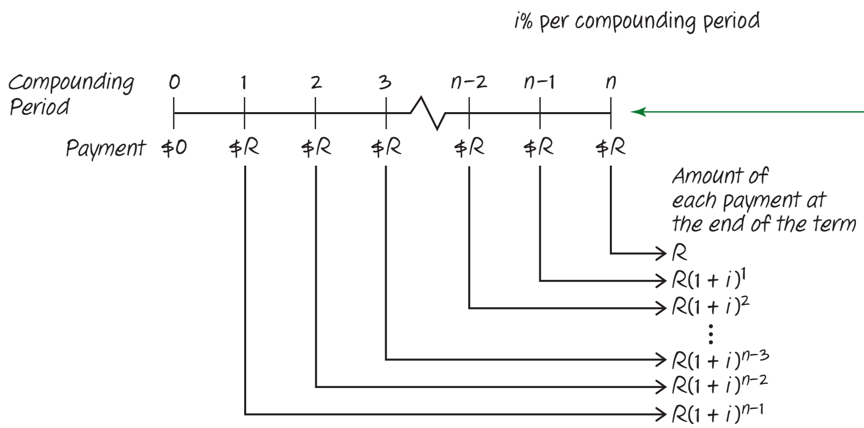
I used the formula for the sum of a geometric series to calculate the total amount of all of Hans's investments.

$$\doteq \$65\ 927.08$$

I rounded to the nearest cent.

The future value of Hans's annuity at the end of 20 years is \$65 927.08.

b)



I drew a timeline of the investments for each compounding period to show the amount of each investment.

The last \$$R$ investment earned no interest. The first \$$R$ investment earned interest $n - 1$ times.

$$R,\ R(1 + i),\ R(1 + i)^2,\ \dots,\ R(1 + i)^{n-2},\ R(1 + i)^{n-1}$$

The values of all of the investments form a geometric sequence with first term R and common ratio $1 + i$.

$$S_n = R + R(1 + i) + R(1 + i)^2 + \dots + R(1 + i)^{n-2} + R(1 + i)^{n-1}$$

The total amount of all these investments is the first n terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{R[(1 + i)^n - 1]}{(1 + i) - 1}$$

$$= R \times \left(\frac{(1 + i)^n - 1}{i}\right)$$

I used the formula for the sum of a geometric series to calculate the total amount of all the investments.

The future value of an annuity in which \$$R$ is invested at the end of each of n regular intervals earning i% of compound interest per interval is

$$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right),$$ where i is expressed as a decimal.

EXAMPLE 2 Selecting a strategy to determine the future value of an annuity

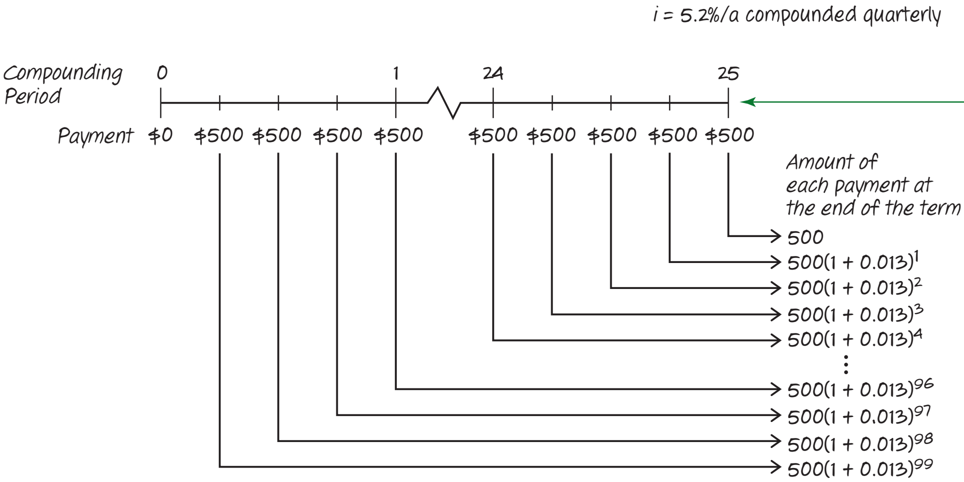
Chie puts away $500 every 3 months at 5.2%/a compounded quarterly. How much will her annuity be worth in 25 years?

Kew's Solution: Using a Geometric Series

$$i = \frac{0.052}{4} = 0.013$$

$$n = 25 \times 4 = 100$$

First I calculated the interest rate per compounding period and the number of compounding periods.



I drew a timeline of the investments for each quarter to show the amounts of each investment. I calculated the value of each investment at the end of 25 years.

$$500, 500(1.013), 500(1.013)^2, \dots, 500(1.013)^{98}, 500(1.013)^{99}$$

The values form a geometric sequence with first term $500 and common ratio 1.013.

$$S_{100} = 500 + 500(1.013) + 500(1.013)^2 + \dots + 500(1.013)^{98} + 500(1.013)^{99}$$

The total amount of all these investments is the first 100 terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{100} = \frac{500(1.013^{100} - 1)}{1.013 - 1}$$

I used the formula for the sum of a geometric series to calculate the total amount of all of Chie's investments.

$$\doteq \$101\ 487.91$$

I rounded to the nearest cent.

The total amount of all of Chie's investments at the end of 25 years will be $101 487.91.

Tina's Solution: Using the Formula for the Future Value of an Annuity

$R = \$500$

$i = \frac{0.052}{4} = 0.013$ ← I calculated the interest rate per compounding period and the number of compounding periods.

$n = 25 \times 4 = 100$

$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$ ← I substituted the values of *R*, *i*, and *n* into the formula for the future value of a simple, ordinary annuity.

$= 500 \times \left(\frac{(1 + 0.013)^{100} - 1}{0.013}\right)$

$\doteq \$101\ 487.91$ ← I rounded to the nearest cent.

The future value of Chie's annuity will be $101 487.91.

EXAMPLE 3 Selecting a strategy to determine the regular payment of an annuity

Sam wants to make monthly deposits into an account that guarantees 9.6%/a compounded monthly. He would like to have $500 000 in the account at the end of 30 years. How much should he deposit each month?

Chantal's Solution

$i = \frac{0.096}{12} = 0.008$ ← I calculated the interest rate per compounding period and the number of compounding periods.

$n = 30 \times 12 = 360$

$FV = \$500\ 000$ ← The future value of the annuity is $500 000.

$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$

$500\ 000 = R \times \left(\frac{(1 + 0.008)^{360} - 1}{0.008}\right)$ ← I substituted the values of *FV*, *i*, and *n* into the formula for the future value of an annuity.

$$500\ 000 \doteq R \times 2076.413$$

$$\frac{500\ 000}{2076.413} = R \times \frac{2076.413}{2076.413}$$

To solve for R, I divided both sides of the equation by 2076.413.

$$R = \$240.80$$

I rounded to the nearest cent.

Sam would have to deposit $240.80 into the account each month in order to have $500 000 at the end of 30 years.

Tech Support

For help using a spreadsheet to enter values and formulas, and to fill down, see Technical Appendix, B-21.

EXAMPLE 4 Selecting a strategy to determine the term of an annuity

Nahid borrows $95 000 to buy a cottage. She agrees to repay the loan by making equal monthly payments of $750 until the balance is paid off. If Nahid is being charged 5.4%/a compounded monthly, how long will it take her to pay off the loan?

Zak's Solution

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$95 000.00
3	1	$750.00	$427.50	$322.50	$94 677.50
4	2	$750.00	"=E3*0.054/12"	"=B4-C4"	"=E3-D4"

I set up a spreadsheet to calculate the balance after every payment. The interest is always charged on the balance and is $\frac{1}{12}$ of 5.4% since it is compounded monthly. The part of the principal that is paid off with each payment is $750, less the interest. The new balance is the old balance, less the part of the principal that is paid.

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$95 000.00
3	1	$750.00	$427.50	$322.50	$94 677.50
4	2	$750.00	$426.05	$323.95	$94 353.55
5	3	$750.00	$424.59	$325.41	$94 028.14
6	4	$750.00	$423.13	$326.87	$93 701.27
7	5	$750.00	$421.66	$328.34	$93 372.92
188	184	$750.00	$16.55	$733.45	$2 944.92
187	185	$750.00	$13.25	$736.75	$2 208.17
188	186	$750.00	$9.94	$740.06	$1 468.11
189	187	$750.00	$6.61	$743.39	$724.72
190	188	$750.00	$3.26	$746.74	−$22.02

I used the FILL DOWN command to complete the spreadsheet until the balance was close to zero.

$$t = \frac{188}{12}$$

$$\doteq 15.667$$

After 188 payments, the balance is close to zero. I calculated the number of years needed to make 188 payments by dividing by 12, since there are 12 payments each year.

$$0.667 \times 12 \text{ months} \doteq 8 \text{ months}$$

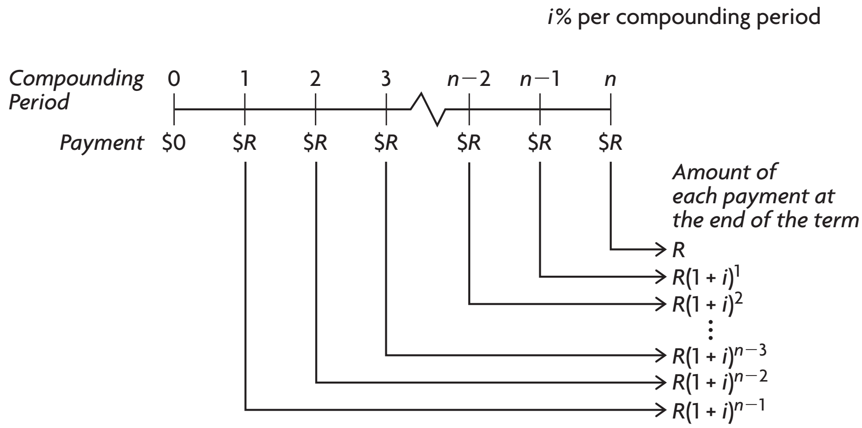
I got a value greater than 15. The 15 meant 15 years, so I had to figure out what 0.667 of a year was.

Nahid can pay off the loan after 188 payments, which would take about 15 years and 8 months.

In Summary

Key Ideas

- The future value of an annuity is the sum of all regular payments and interest earned.



- The future value can be written as the geometric series

$$FV = R + R(1 + i) + R(1 + i)^2 + \ldots + R(1 + i)^{n-2} + R(1 + i)^{n-1}$$

  where FV is the future value; R is the regular payment; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.
- The formula for the sum of a geometric series can be used to determine the future value of an annuity.

Need to Know

- A variety of technological tools (spreadsheets, graphing calculators) can be used to solve problems involving annuities.
- The payment interval of an annuity is the time between successive payments.
- The term of an annuity is the time from the first payment to the last payment.
- The formula for the future value of an annuity is

$$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$$

  where FV is the future value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

CHECK Your Understanding

1. Each year, Eric invests $2500 at 8.2%/a compounded annually for 25 years.
   a) Calculate the value of each of the first four investments at the end of 25 years.
   b) What type of sequence do the values form?
   c) Determine the total amount of all of Eric's investments.

2. Calculate the future value of each annuity.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	$100 per month	3.6%	monthly	50 years
b)	$1500 per quarter	6.2%	quarterly	15 years
c)	$500 every 6 months	5.6%	semi-annually	8 years
d)	$4000 per year	4.5%	annually	10 years

3. Lois invests $650 every 6 months at 4.6%/a compounded semi-annually for 25 years. How much interest will she earn after the 25th year?

4. Josh borrows some money on which he makes monthly payments of $125.43 for 3 years. If the interest rate is 5.4%/a compounded monthly, what will be the total amount of all of the payments at the end of the 3 years?

PRACTISING

5. K Calculate the future value of each annuity.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	$1500 per year	6.3%	annually	10 years
b)	$250 every 6 months	3.6%	semi-annually	3 years
c)	$2400 per quarter	4.8%	quarterly	7 years
d)	$25 per month	8%	monthly	35 years

6. A Mike wants to invest money every month for 40 years. He would like to have $1 000 000 at the end of the 40 years. For each investment option, how much does he need to invest each month?
   a) 10.2%/a compounded monthly
   b) 5.1%/a compounded monthly

7. Kiki has several options for investing $1200 per year:

	Regular Payment	Rate of Compound Interest per Year	Compounding Period
a)	$100 per month	7.2%	monthly
b)	$300 per quarter	7.2%	quarterly
c)	$600 every 6 months	7.2%	semi-annually
d)	$1200 per year	7.2%	annually

Without doing any calculations, which investment would be best? Justify your reasoning.

8. Kenny wants to invest $250 every three months at 5.2%/a compounded quarterly. He would like to have at least $6500 at the end of his investment. How long will he need to make regular payments?

9. Sonja and Anita want to make equal monthly payments for the next 35 years. At the end of that time, each person would like to have $500 000. Sonja's bank will give her 6.6%/a compounded monthly. Anita can invest through her work and earn 10.8%/a compounded monthly.

a) How much more per month does Sonja have to invest?

b) If Anita decides to invest the same monthly amount as Sonja, how much more money will she have at the end of 35 years?

10. **T** Jamal wants to invest $150 every month for 10 years. At the end of that time, he would like to have $25 000. At what annual interest rate, compounded monthly, does Jamal need to invest to reach his goal? Round your answer to two decimal places.

11. **C** Draw a mind map for the concept of *future value of annuities*. Show how it is related to interest, sequences, and series.

Extending

12. Carmen borrows $10 000 at 4.8%/a compounded monthly. She decides to make monthly payments of $250.

a) How long will it take her to pay off the loan?

b) How much interest will she pay over the term of the loan?

13. Greg borrows $123 000 for the purchase of a house. He plans to make regular monthly payments over the next 20 years to pay off the loan. The bank is charging Greg 6.6%/a compounded monthly. What monthly payments will Greg have to make?

14. How many equal monthly payments would you have to make to get 100 times the amount you are investing each month if you are earning 8.4%/a compounded monthly?

8.5 Annuities: Present Value

GOAL

Determine the present value of an annuity earning compound interest.

YOU WILL NEED

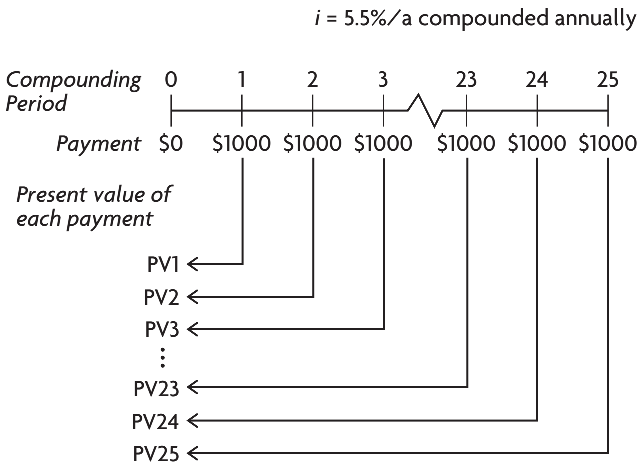
- graphing calculator
- spreadsheet software

INVESTIGATE the Math

Kew wants to invest some money at 5.5%/a compounded annually. He would like the investment to provide $1000 for scholarships at his old high school at the end of each year for the next 25 years.

? How much should Kew invest now?

A. Copy the timeline shown. How would you calculate each of the present values PV1 to PV25?



B. How much would Kew need to invest now if he wanted to provide $1000 at the end of the 1st year?

C. How much would Kew need to invest now if he wanted to provide $1000 at the end of the 2nd, 3rd, and 4th years, respectively?

D. How is the present value after 2 years (PV2) related to the present value after 1 year (PV1)?

E. Set up a spreadsheet with columns as shown at the right. Enter your values of PV1 and PV2 in the Present Value column.

	A	B	C
1	Year	Scholarship Payment	Present Value
2	1	$1 000.00	
3	2	$1 000.00	
4	3	$1 000.00	
5	4	$1 000.00	
6	5	$1 000.00	
7	6	$1 000.00	
8	7	$1 000.00	
9	8	$1 000.00	
10	9	$1 000.00	
11	10	$1 000.00	

F. Use the relationship among the present values to complete the rest of the entries under Present Value.

G. Use the values in the Present Value column to determine how much Kew would need to invest now in order to provide the scholarships for the next 25 years.

Reflecting

H. What type of sequence do the present values in part F form?

I. Describe a method that you could have used to solve this problem without using a spreadsheet.

APPLY the Math

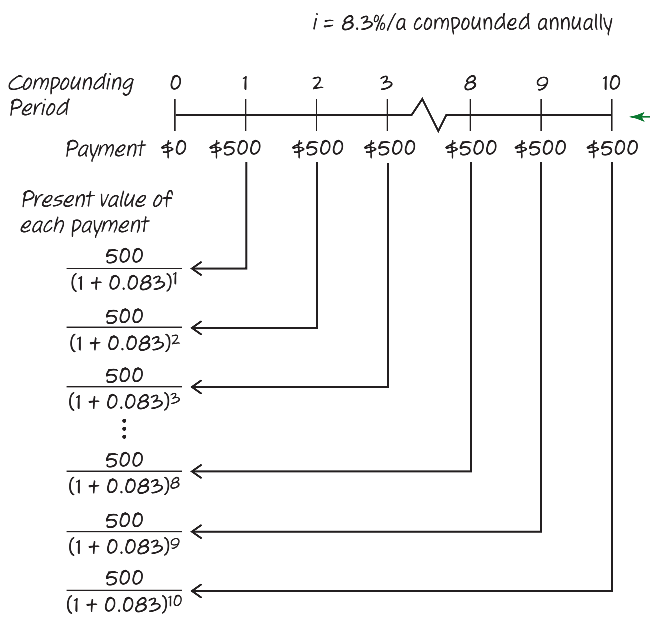
EXAMPLE 1 Representing the present value of an annuity earning compound interest as a series

a) How much would you need to invest now at 8.3%/a compounded annually to provide $500 per year for the next 10 years?

b) How much would you need to invest now to provide n regular payments of $\$R$ if the money is invested at a rate of i% per compounding period?

Tara's Solution

a)



I drew a timeline showing the $500 payments for the next 10 years.

$$PV = \frac{A}{(1 + i)^n}$$

I considered each payment separately and used the present-value formula to determine how much would need to be invested now to provide each $500 payment.

$$PV_1 = \frac{500}{(1.083)}$$

$$PV_2 = \frac{500}{(1.083)^2}$$

$$PV_3 = \frac{500}{(1.083)^3}$$

$$\vdots$$

$$PV_9 = \frac{500}{(1.083)^9}$$

$$PV_{10} = \frac{500}{(1.083)^{10}}$$

$$a = \frac{500}{(1.083)} = 500 \times 1.083^{-1}$$

$$r = \frac{1}{1.083} = 1.083^{-1}$$

$$n = 10$$

The present values for each payment are the first 10 terms of a geometric sequence with first term 500×1.083^{-1} and common ratio 1.083^{-1}.

$$S_{10} = 500 \times 1.083^{-1} + 500 \times 1.083^{-2} + 500 \times 1.083^{-3} + \ldots + 500 \times 1.083^{-9} + 500 \times 1.083^{-10}$$

The total amount of money invested now has to provide each of the \$500 future payments. So I had to calculate the sum of all of the present values.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

The sum of the present values forms a geometric series, so I used the formula for the sum of a geometric series.

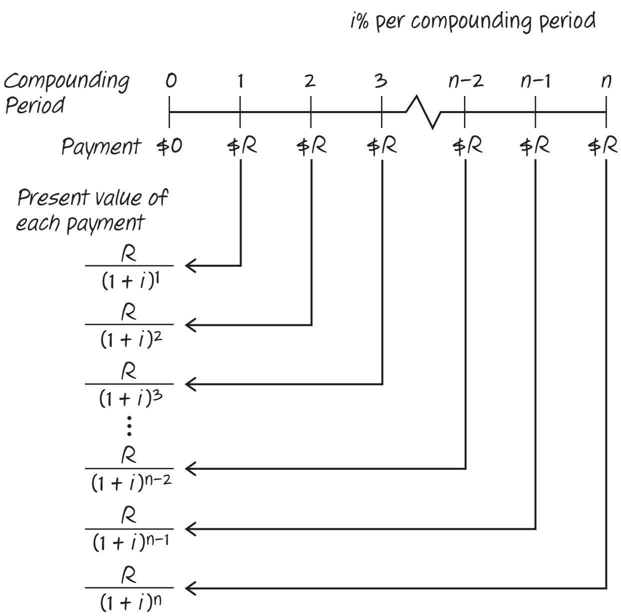
$$S_{10} = \frac{500 \times 1.083^{-1}[(1.083^{-1})^{10} - 1]}{1.083^{-1} - 1}$$

$$\doteq \$3310.11$$

I rounded to the nearest cent.

A sum of \$3310.11 invested now would provide a payment of \$500 for each of the next 10 years.

b)



I drew a timeline showing the $\$R$ payments for n regular intervals.

$$PV = \frac{A}{(1 + i)^n}$$

I considered each payment separately and used the present-value formula to determine how much would need to be invested now to provide each specific $*R* payment.

$$PV_1 = \frac{R}{1 + i}$$

$$PV_2 = \frac{R}{(1 + i)^2}$$

$$PV_3 = \frac{R}{(1 + i)^3}$$

$$\vdots$$

$$PV_n = \frac{R}{(1 + i)^n}$$

$$a = R \times (1 + i)^{-1}$$

$$r = (1 + i)^{-1}$$

I used negative exponents, since I was dividing by $1 + i$ each time.

The present values for each payment are the first n terms of a geometric sequence with first term $R \times (1 + i)^{-1}$ and common ratio $(1 + i)^{-1}$.

$$S_n = R \times (1 + i)^{-1} + R \times (1 + i)^{-2} + R \times (1 + i)^{-3} + \ldots + R \times (1 + i)^{-n}$$

I needed to determine the total amount of money invested now to provide each of the $*R* future payments. So I had to calculate the sum of all of the present values.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

The sum of the present values forms a geometric series, so I used the formula for the sum of a geometric series.

$$= \frac{R \times (1 + i)^{-1}[(1 + i)^{-n} - 1]}{(1 + i)^{-1} - 1}$$

$$= \frac{R \times (1 + i)^{-1}[(1 + i)^{-n} - 1]}{(1 + i)^{-1} - 1} \times \frac{1 + i}{1 + i}$$

The numerator and denominator each have a factor of $(1 + i)^{-1}$, so I multiplied them both by $1 + i$ to simplify.

$$= \frac{R[(1 + i)^{-n} - 1]}{1 - (1 + i)}$$

$$= \frac{R[(1 + i)^{-n} - 1]}{-i}$$

$$= R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$

I multiplied the numerator and denominator by -1 to simplify.

The present value of an annuity in which $*R* is paid at the end of each of n regular intervals earning i% compound interest per interval is $PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$.

EXAMPLE 2 Selecting a strategy to determine the present value of an annuity

Sharon won a lottery that offers \$50 000 a year for 20 years or a lump-sum payment now. If she can invest the money at 5%/a compounded annually, how much should the lump-sum payment be to be worth the same amount as the annuity?

Joel's Solution: Using a Spreadsheet

	A	B	C
1	Year	Payment	Present Value
2	1	\$50 000.00	"= B2/1.05"
3	2	\$50 000.00	"= B3/(1.05)^A3"
4	3	\$50 000.00	"= B4/(1.05)^A4"

I set up a spreadsheet to determine the present value of each of the payments for the next 20 years. The present value of each payment is given by the formula $PV = \frac{A}{(1 + i)^n}$, so the present value of the payments form a geometric sequence with $r = \frac{1}{1 + i}$. Since Sharon is earning 5%/a, the present value of each following year is equal to 1.05 times the present value of the previous year.

	A	B	C
1	Year	Payment	Present Value
2	1	\$50 000.00	\$47 619.05
3	2	\$50 000.00	\$45 351.47
4	3	\$50 000.00	\$43 191.88
5	4	\$50 000.00	\$41 135.12
6	5	\$50 000.00	\$39 176.31
7	6	\$50 000.00	\$37 310.77
8	7	\$50 000.00	\$35 534.07
9	8	\$50 000.00	\$33 841.97
10	9	\$50 000.00	\$32 230.45
11	10	\$50 000.00	\$30 695.66
12	11	\$50 000.00	\$29 233.96
13	12	\$50 000.00	\$27 841.87
14	13	\$50 000.00	\$26 516.07
15	14	\$50 000.00	\$25 253.40
16	15	\$50 000.00	\$24 050.85
17	16	\$50 000.00	\$22 905.58
18	17	\$50 000.00	\$21 814.83
19	18	\$50 000.00	\$20 776.03
20	19	\$50 000.00	\$19 786.70
21	20	\$50 000.00	\$18 844.47
22			\$623 110.52

I used the FILL DOWN command to determine the present values for the remaining payments. I then used the SUM command to determine the sum of all the present values.

The lump-sum payment should be \$623 110.52.

EXAMPLE 3 Selecting a strategy to determine the regular payment and total interest of an annuity

Len borrowed \$200 000 from the bank to purchase a yacht. If the bank charges 6.6%/a compounded monthly, he will take 20 years to pay off the loan.

a) How much will each monthly payment be?
b) How much interest will he have paid over the term of the loan?

Jasmine's Solution: Using the Formula

a)

$$i = \frac{0.066}{12} = 0.0055$$

$$n = 20 \times 12 = 240$$

$$PV = \$200\,000$$

I calculated the interest rate per compounding period and the number of compounding periods.

$$PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$

$$200\,000 = R \times \left(\frac{1 - (1 + 0.0055)^{-240}}{0.0055}\right)$$

I substituted the values of *PV*, *i*, and *n* into the formula for the present value of an annuity.

$$200\,000 \doteq R \times 133.072$$

$$\frac{200\,000}{133.072} = R \times \frac{133.072}{133.072}$$

To solve for *R*, I divided both sides of the equation by 133.072.

$$R \doteq 1502.94$$

I rounded to the nearest cent.

Len will have to pay \$1502.94 per month for 20 years to pay off the loan.

b)

$$A = 1502.94 \times 240$$
$$= \$360\,706.60$$

I calculated the total amount that Len will have paid over the 20-year term.

$$I = A - PV$$
$$= \$360\,706.60 - \$200\,000$$
$$= \$160\,706.60$$

I determined the interest by subtracting the present value from the total amount that Len will have paid.

Over the 20-year term of the loan, Len will have paid \$160 706.60 in interest.

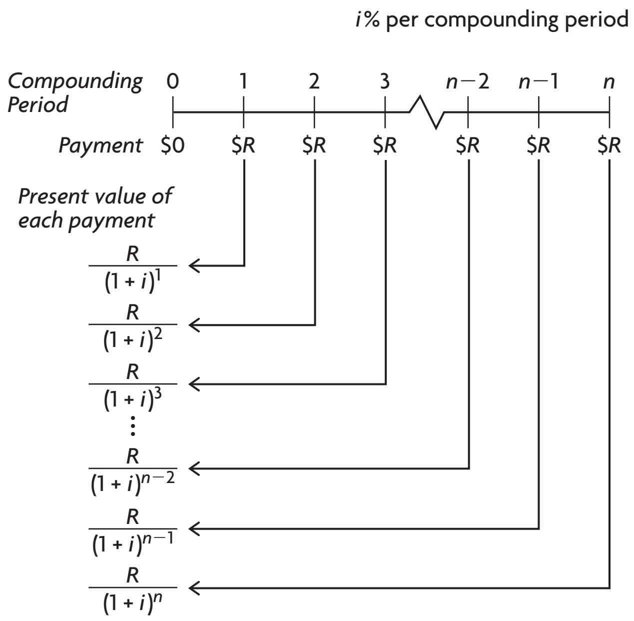
In Summary

Key Ideas

- The present value of an annuity is the value of the annuity at the beginning of the term. It is the sum of all present values of the payments and can be written as the geometric series

$$PV = R \times (1 + i)^{-1} + R \times (1 + i)^{-2} + R \times (1 + i)^{-3} + \ldots + R \times (1 + i)^{-n}$$

where PV is the present value; R is the regular payment; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.



- The formula for the sum of a geometric series can be used to determine the present value of an annuity.

Need to Know

- The formula for the present value of an annuity is

$$PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$

where PV is the present value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

CHECK Your Understanding

1. Each situation represents a loan.

i) Draw a timeline to represent the amount of the original loan.
ii) Write the series that represents the amount of the original loan.
iii) Calculate the amount of the original loan.
iv) Calculate the interest paid.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$650 per year	3.7%	annually	5 years
b)	\$1200 every 6 months	9.4%	semi-annually	9 years
c)	\$84.73 per quarter	3.6%	quarterly	$3\frac{1}{2}$ years
d)	\$183.17 per month	6.6%	monthly	10 years

2. Each situation represents a simple, ordinary annuity.

i) Calculate the present value of each payment.
ii) Write the present values of the payments as a series.
iii) Calculate the present value of the annuity.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$8000 per year	9%	annually	7 years
b)	\$300 every 6 months	8%	semi-annually	3.5 years
c)	\$750 per quarter	8%	quarterly	2 years

PRACTISING

3. Calculate the present value of each annuity.
K

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$5000 per year	7.2%	annually	5 years
b)	\$250 every 6 months	4.8%	semi-annually	12 years
c)	\$25.50 per week	5.2%	weekly	100 weeks
d)	\$48.50 per month	23.4%	monthly	$2\frac{1}{2}$ years

4. You want to buy a $1300 stereo on credit and make monthly payments over 2 years. If the store is charging you 18%/a compounded monthly, what will be your monthly payments?

5. Lily wants to buy a snowmobile. She can borrow $7500 at 10%/a compounded quarterly if she repays the loan by making equal quarterly payments for 4 years.
   a) Draw a timeline to represent the annuity.
   b) Write the series that represents the present value of the annuity.
   c) Calculate the quarterly payment that Lily must make.

6. Rocco pays $50 for a DVD/CD player and borrows the remaining amount. He plans to make 10 monthly payments of $40 each. The first payment is due next month.
   a) The interest rate is 18%/a compounded monthly. What was the selling price of the player?
   b) How much interest will he have paid over the term of the loan?

7. Emily is investing $128 000 at 7.8%/a compounded monthly. She wants to withdraw an equal amount from this investment each month for the next 25 years as spending money. What is the most she can take out each month?

8. The Peca family wants to buy a cottage for $69 000. The Pecas can pay $5000 and finance the remaining amount with a loan at 9%/a compounded monthly. The loan payments are monthly, and they may choose either a 7-year or a 10-year term.
   a) Calculate the monthly payment for each term.
   b) How much would they save in interest by choosing the shorter term?
   c) What other factors should the Pecas consider before making their financing decision?

9. **A** Charles would like to buy a new car that costs $32 000. The dealership offers to finance the car at 2.4%/a compounded monthly for five years with monthly payments. The dealer will reduce the selling price by $3000 if Charles pays cash. Charles can get a loan from his bank at 5.4%/a compounded monthly. Which is the best way to buy the car? Justify your answer with calculations.

10. To pay off $35 000 in loans, Nina's bank offers her a rate of 8.4%/a compounded monthly. She has a choice between a 5-, 10-, or 15-year term.
    a) Determine the monthly payment for each term.
    b) Calculate how much interest Nina would pay in each case.

11. Pedro pays $45 for a portable stereo and borrows the remaining amount. The loan payments are $25 per month for 1 year. The interest rate is 18.6%/a compounded monthly.
    a) What was the selling price of the stereo?
    b) How much interest will Pedro have paid over the term of the loan?

12. Suzie buys a new computer for $2500. She pays $700 and finances the rest at $75.84 per month for $2\frac{1}{2}$ years. What annual interest rate, compounded monthly, is Suzie being charged? Round your answer to two decimal places.

13. Leo invests $50 000 at 11.2%/a compounded quarterly for his retirement. Leo's financial advisor tells him that he should take out a regular amount quarterly when he retires. If Leo has 20 years until he retires and wants to use the investment for recreation for the first 10 years of retirement, what is the maximum quarterly withdrawal he can make?

14. Charmaine calculates that she will require about $2500 per month for the first 15 years of her retirement. If she has 25 years until she retires, how much should she invest each month at 9%/a compounded monthly for the next 25 years if she plans to withdraw $2500 per month for the 15 years after that?

15. **T** A lottery has two options for winners collecting their prize:
- Option A: $1000 each week for life
- Option B: $660 000 in one lump sum

The current interest rate is 6.76%/a compounded weekly.

a) Which option would you suggest to a winner who expects to live for another 25 years?

b) When is option A better than option B?

16. **C** Classify situations and factors that show the differences between each pair of terms. Give examples.

a) a lump sum or an annuity

b) future value or present value

Extending

17. Stefan claims that he has found a different method for calculating the present value of an annuity. Instead of calculating the present value of each payment, he calculates the future value of each payment. Then he calculates the sum of the future values of the payments. Finally, he calculates the present value of this total sum.

a) Use Stefan's method to solve Example 1 (a).

b) Create another example to show that his claim is true. Include timelines.

c) Use the formula for present value to prove that Stefan's claim works for all annuities.

18. Kyla must repay student loans that total $17 000. She can afford to make $325 monthly payments. The bank is charging an interest rate of 7.2%/a compounded monthly. How long will it take Kyla to repay her loans?

19. In question 14, Charmaine invested a fixed amount per month so that her annuity would provide her with another monthly amount in her retirement. Derive a formula for the regular payment $\$R$ that must be made for m payments at an interest rate of $i\%$ per compounding period to provide for a regular withdrawal of $\$W$ after all the payments are made for n withdrawals.

8.6 Using Technology to Investigate Financial Problems

GOAL

Use technology to investigate the effects of changing the conditions in financial problems.

YOU WILL NEED

- graphing calculator
- spreadsheet software

INVESTIGATE the Math

Tina wants to buy a stereo that costs $566.47 after taxes. The store allows her to buy the stereo by making payments of $29.95 per month for 2 years.

? What annual interest rate, compounded monthly, is the store charging?

A. Draw a timeline for this situation. Will you be calculating present values or future values?

B. Use a spreadsheet to set up an **amortization schedule** as shown.

	A	B	C	D	E	F
1	**Interest Rate**	**Payment Number**	**Payment**	**Interest Paid**	**Principal Paid**	**Balance**
2						$566.47
3	0.01	1	$29.95	"=F2*A3"	"=C3-D3"	"=F2-E3"
4		"=B3+1"	$29.95	"=F3*A3"	"=C4-D4"	"=F3-E4"

amortization schedule

a record of payments showing the interest paid, the principal, and the current balance on a loan or investment

C. Use the COPY DOWN command to complete the spreadsheet so that 24 payments are showing. The spreadsheet shown here is set up with an interest rate of 1% per compounding period. Adjust the value of the interest rate to solve the problem.

Tech Support

For help creating an amortization schedule using a spreadsheet, see Technical Appendix, B-22.

D. Enter the formula for the present value of the annuity into a graphing calculator, where Y is the (unknown) present value and X is the annual interest rate compounded monthly.

E. Graph the equation in part D, as well as $y = 566.47$. Use these graphs to solve the problem.

F. On your graphing calculator, activate the TVM Solver.

G. Enter the corresponding values and then solve the problem.

Tech Support

For help using the TVM Solver on a graphing calculator, see Technical Appendix, B-19.

Reflecting

H. Why could you not solve this problem easily with pencil and paper?

I. Which of the three methods (the spreadsheet in parts B and C, the graphs in parts D and E, or the TVM Solver in parts F and G) used to solve the problem do you prefer? Why?

APPLY the Math

EXAMPLE 1 Selecting a tool to investigate the effects of varying the interest rate

Jamal has $10 000 to invest. Bank of North America offers an interest rate of 4.2%/a compounded monthly. Key Bank offers an interest rate of 5%/a compounded quarterly. How much longer will it take the money invested to grow to $50 000 if Jamal chooses Bank of North America?

Lina's Solution: Using Guess-and-Check

$$i = \frac{0.042}{12} = 0.0035$$

I first looked at the Bank of North America. Since interest is paid monthly, I divided the annual interest rate by 12 to get the interest rate per month.

$$P = \$10\,000$$

$$A = P(1 + i)^n$$
$$= 10\,000(1.0035)^n$$

I substituted the values of i and P into the compound-interest formula. I thought 10 years might be a good guess. That would give $n = 120$ compounding periods.

$$A = 10\,000(1.0035)^{120}$$
$$\doteq 15\,208.46$$

$$A = 10\,000(1.0035)^{480}$$
$$\doteq 53\,498.41$$

My guess was way too small, so I tried 40 years, which gives $n = 480$ compounding periods.

Try $n = 460$:

$$A = 10\,000(1.0035)^{460}$$
$$\doteq 49\,887.68$$

Try $n = 461$:

$$A = 10\,000(1.0035)^{461}$$
$$\doteq 50\,062.29$$

That guess was much closer. Eventually, I tried 460 months. It was a little low, so I tried 461 months.

$$\frac{461}{12} \doteq 38.417$$

I determined how long 461 months is in terms of years. First I divided 461 by 12 to get 38 years.

$$0.417 \times 12 \text{ months} \doteq 5 \text{ months}$$

Then I multiplied 0.417 by 12 to get 5 months.

$$n = 38 \text{ years and } 5 \text{ months}$$

$$i = \frac{0.05}{4} = 0.0125$$

Next, I looked at Key Bank. Since interest is paid quarterly, I divided the annual interest rate by 4 to get the interest rate per quarter.

$$P = \$10\,000$$

$$A = P(1 + i)^n$$
$$= 10\,000(1.0125)^n$$

I substituted the values of i and P into the compound-interest formula.

$$A = 10\,000(1.0125)^{140}$$
$$\doteq 56\,925.19$$

Since it took Bank of North America 38 years to grow to $50 000, I used 35 years as my first guess for Key Bank because the interest rate is higher. 35 years is $35 \times 4 = 140$ quarters.

Try $n = 129$: $A = 10\,000(1.0125)^{129} \doteq 49\,654.56$

Try $n = 130$: $A = 10\,000(1.0125)^{130} \doteq 50\,275.24$

This result was close, but a bit high. Eventually, I tried 129 quarters and then 130 quarters.

$$\frac{130}{4} = 32.5$$

I determined how long 130 quarters is in terms of years. I divided 130 by 4 to get 32 years.

$n =$ 32 years and 6 months

I knew that 0.5 years is 6 months.

It will take 38 years and 5 months to get $50 000 if Jamal chooses Bank of North America. It will take 32 years and 6 months if he chooses Key Bank. So it will take almost 6 years longer to reach his goal if Jamal chooses Bank of North America.

George's Solution: Using a Graphing Calculator

$A = P(1 + i)^n$

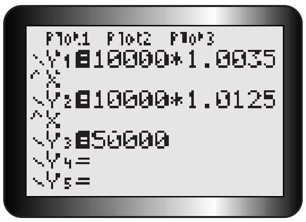
Bank of North America: $i = \frac{0.042}{12} = 0.0035$

Key Bank: $i = \frac{0.05}{4} = 0.0125$

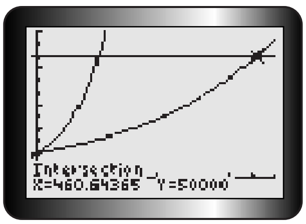
At Bank of North America, interest is compounded monthly. At Key Bank, interest is compounded quarterly. I calculated the interest rate per compounding period at each bank.

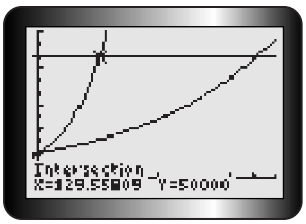
$A = 10\,000(1.0035)^n$ $A = 10\,000(1.0125)^n$

Then I used the compound-interest formula to calculate the amounts.



I entered the equations for the amounts into my graphing calculator, using Y1 and Y2 for the amounts for Bank of North America and Key Bank, respectively, and X for the number of compounding periods. I entered Y3 = 50 000.





I graphed the three equations and used the calculator to find the point of intersection of each exponential function with the horizontal line, which indicated when the amount of the investment had reached $50 000.

It will take about 460 months, or 38 years and 5 months, to get $50 000 if Jamal chooses Bank of North America. It will take about 129 quarters, or 32 years and 6 months, if he chooses Key Bank. So it will take almost 6 years longer to reach his goal if Jamal chooses Bank of North America.

Coco's Solution: Using the TVM Solver

I entered the information on the investment with Bank of North America into the TVM Solver. Jamal pays into the account at the start, so the present value is −$10 000. Also, no regular payments are being made. This is a lump-sum investment, so I set PMT on my calculator to 0. I determined that it would take a bit more than 460 months, or 38 years and 5 months, to reach his goal with Bank of North America.

I entered the information on the investment with Key Bank into the TVM Solver. I determined that it would take a bit more than 129 quarters, or 32 years and 6 months, to reach his goal with Key Bank.

If Jamal chooses Bank of North America, it will take about 6 years longer to reach his goal.

EXAMPLE 2 Selecting a tool to investigate the effects of increasing the monthly payment

Lia borrows $180 000 to open a restaurant. She can afford to make monthly payments between $1000 and $1500 at 4.8%/a compounded monthly. How much sooner can she pay off the loan if she makes the maximum monthly payment?

Teresa's Solution: Using a Spreadsheet

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$180 000.00
3	1	$1 000 .00	"=E2*(0.048/12)"	"=B3-C3"	"=E2-D3"
4	"=A3+1"	$1 000 .00	"=E3*(0.048/12)"	"=B4-C4"	"=E3-D4"

I set up a spreadsheet to solve the problem. Since the interest is compounded monthly, I divided 4.8% by 12 to get the interest rate per month. For the $1000 minimum payment, I calculated the proportion of the principal paid for each payment. Then I subtracted that proportion from the previous balance to get the balance at the end of the next month.

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$180 000.00
3	1	$1 000 .00	$720.00	$280.00	$179 720.00
4	2	$1 000 .00	$718.88	$281.12	$179 438.88
5	3	$1 000 .00	$717.76	$282.24	$179 156.64
6	4	$1 000 .00	$716.63	$283.37	$178 873.26
7	5	$1 000 .00	$715.49	$284.51	$178 588.76
8	6	$1 000 .00	$714.36	$285.64	$178 303.11
9	7	$1 000 .00	$713.21	$286.79	$178 016.32
10	8	$1 000 .00	$712.07	$287.93	$177 728.39
11	9	$1 000 .00	$710.91	$289.09	$177 439.30
12	10	$1 000 .00	$709.76	$290.24	$177 149.06

Next, I used the FILL DOWN command to complete the other rows.

314	312	$1 000 .00	$30.96	$969.04	$6 770.39
315	313	$1 000 .00	$27.08	$972.92	$5 797.48
316	314	$1 000 .00	$23.19	$976.81	$4 820.67
317	315	$1 000 .00	$19.28	$980.72	$3 839.95
318	316	$1 000 .00	$15.36	$984.64	$2 855.31
319	317	$1 000 .00	$11.42	$988.58	$1 866.73
320	318	$1 000 .00	$7.47	$992.53	$874.20
321	319	$1 000 .00	$3.50	$996.50	–$122.31

I continued until the balance became negative, indicating that the loan was paid off.

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$180 000.00
3	1	$1 500 .00	$720.00	$780.00	$179 220.00
4	2	$1 500 .00	$716.88	$783.12	$178 436.88
5	3	$1 500 .00	$713.75	$786.25	$177 650.63
6	4	$1 500 .00	$710.60	$789.40	$176 861.23
7	5	$1 500 .00	$707.44	$792.56	$176 068.67
8	6	$1 500 .00	$704.27	$795.73	$175 272.95
9	7	$1 500 .00	$701.09	$798.91	$174 474.04
10	8	$1 500 .00	$697.90	$802.10	$173 671.94
11	9	$1 500 .00	$694.69	$805.31	$172 866.63
12	10	$1 500 .00	$691.47	$808.53	$172 058.09

I replaced the $1000 minimum payment with the $1500 maximum payment and used the FILL DOWN command in all the cells under Payment.

159	157	$1 500 .00	$46.04	$1 453.96	$10 054.91
160	158	$1 500 .00	$40.22	$1 459.78	$8 595.13
161	159	$1 500 .00	$34.38	$1 465.62	$7 129.51
162	160	$1 500 .00	$28.52	$1 471.48	$5 658.03
163	161	$1 500 .00	$22.63	$1 477.37	$4 180.66
164	162	$1 500 .00	$16.72	$1 483.28	$2 697.39
165	163	$1 500 .00	$10.79	$1 489.21	$1 208.17
166	164	$1 500 .00	$4.83	$1 495.17	–$286.99

I continued until the balance became negative, indicating that the loan was paid off.

At the minimum payment of $1000, Lia's loan would be paid off after 319 months, or 26 years and 7 months. At the maximum payment of $1500, the loan would be paid off after 164 months, or 13 years and 8 months. So Lia can pay off the loan almost 13 years sooner if she makes the maximum payment.

Mike's Solution: Using the TVM Solver

I entered the information on the loan into the TVM Solver on a graphing calculator. I entered the minimum monthly payment of $1000 and then used the calculator to determine the number of payments needed.

I then changed the monthly payment to the maximum amount of $1500, and used the calculator to determine the number of payments needed.

At the minimum payment of $1000, Lia's loan would be paid off after 319 months, or 26 years and 7 months. At the maximum payment of $1500, the loan would be paid off after 164 months, or 13 years and 8 months. So Lia can pay off the loan almost 13 years sooner if she makes the maximum payment.

Communication | ***Tip***

Sometimes you can make a large purchase by paying a small portion of the cost right away and financing the rest with a loan. The portion paid right away is called a **down payment**.

EXAMPLE 3 Selecting a tool to investigate the effects of paying more frequently

Sarah and John are both saving for a down payment on their first home. Both plan to save $2400 each year by depositing into an account that earns 4%/a.

- John makes monthly deposits of $200 into an account on which the interest is compounded monthly.
- Sarah makes annual payments of $2400 into an account on which the interest is compounded annually.

Determine the difference in their account balances at the end of 5 years.

Jason's Solution

I used the TVM Solver on my graphing calculator and entered the information on John. I found that his balance would be $13 259.80 at the end of 5 years.

I repeated the same type of calculation, but this time with the information on Sarah. I found that her balance would be $12 999.17 at the end of 5 years.

$\$13\ 259.80 - \$12\ 999.17 = \$260.63$

I subtracted to calculate the difference in the amounts.

John's account will have $260.63 more than Sarah's after 5 years.

EXAMPLE 4 Connecting the interest paid on a loan with time

You borrow $100 000 at 8.4%/a compounded monthly. You make monthly payments of $861.50 to pay off the loan after 20 years. How long will it take to pay off

a) the first $25 000?
b) the next $25 000?
c) the next $25 000?
d) the last $25 000?
e) Why are the answers to parts (a) through (d) all different?

Mena's Solution

a)

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$100 000.00
3	1	$861.50	$700.00	$161.50	$99 838.50
4	2	$861.50	$698.87	$162.63	$99 675.87
5	3	$861.50	$697.73	$163.77	$99 512.10
6	4	$861.50	$696.58	$164.92	$99 347.19
7	5	$861.50	$695.43	$166.07	$99 181.12
104	102	$861.50	$534.80	$326.70	$76 073.31
105	103	$861.50	$532.51	$328.99	$75 744.32
106	104	$861.50	$530.21	$331.29	$75 413.03
107	105	$861.50	$527.89	$333.61	$75 079.42
108	106	$861.50	$525.56	$335.94	$74 743.48

I used a spreadsheet to create an amortization schedule. I then used the FILL DOWN feature to complete the spreadsheet.

I noticed that the balance is reduced to $74 743.48 after 106 months, so it took 8 years and 10 months to pay off the first $25 000.

b)

164	162	$861.50	$365.00	$496.50	$51 646.68
165	163	$861.50	$361.53	$499.97	$51 146.71
166	164	$861.50	$358.03	$503.47	$50 643.23
167	165	$861.50	$354.50	$507.00	$50 136.24
168	166	$861.50	$350.95	$510.55	$49 625.69

The balance is reduced to $49 625.69 after 166 months, so it took 60 months, or 5 years, to pay off the next $25 000.

c)

206	204	$861.50	$195.99	$665.51	$27 332.96
207	205	$861.50	$191.33	$670.17	$26 662.79
208	206	$861.50	$186.64	$674.86	$25 987.93
209	207	$861.50	$181.92	$679.58	$25 308.35
210	208	$861.50	$177.16	$684.34	$24 624.00

The balance is reduced to $24 624.00 after 208 months, so it took 42 months, or 3 years and 6 months, to pay off the next $25 000.

d)

239	237	$861.50	$23.72	$837.78	$2 551.46
240	238	$861.50	$17.86	$843.64	$1 707.82
241	239	$861.50	$11.95	$849.55	$858.28
242	240	$861.50	$6.01	$855.49	$2.78
243	241	$861.50	$0.02	$861.48	–$858.70

The loan is paid off after 240 months, or 20 years. It takes 208 months to pay about $75 000, so I subtracted 208 from 240 to determine how long it takes to pay the last $25 000 of the loan. The last $25 000 takes 32 months, or 2 years and 8 months, to pay off.

e) It takes different lengths of time to pay off the same amount of money because the interest paid is greater when the balance owed is greater. Less of the payment goes toward the principal.

In Summary

Key Idea

- Spreadsheets and graphing calculators are just two of the technological tools that can be used to investigate and solve financial problems involving interest, annuities, and amortization schedules.

Need to Know

- The advantage of an amortization schedule is that it provides the history of all payments, interest paid, and balances on a loan.
- More interest can be earned if
  - the interest rate is higher
  - there are more compounding periods per year
- If you increase the amount of the regular payment of a loan, you can pay it off sooner and save a significant amount in interest charges.
- Early in the term of a loan, the major proportion of each regular payment is interest, with only a small amount going toward paying off the principal. As time progresses, a larger proportion of each regular payment goes toward the principal.

CHECK Your Understanding

1. Use technology to determine how long it will take to reach each investment goal.

	Principal	Rate of Compound Interest per Year	Compounding Period	Future Value
a)	$5 000	8.3%	annually	$13 000
b)	$2 500	6.8%	semi-annually	$4 000
c)	$450	12.4%	quarterly	$4 500
d)	$15 000	3.6%	monthly	$20 000

2. Use technology to determine the annual interest rate, to two decimal places, being charged in each loan. The compounding period corresponds to when the payments are made.

	Principal	Regular Payment	Time
a)	$2 500	$357.59 per year	10 years
b)	$15 000	$1497.95 every 6 months	6 years
c)	$3 500	$374.56 per quarter	3 years
d)	$450	$29.62 per month	18 months

PRACTISING

3. Trevor wants to save $3500. How much will he have to put away each month at 12.6%/a compounded monthly in order to have enough money in $2\frac{1}{2}$ years?

4. **A** Nadia borrows $120 000 to buy a house. The current interest rate is 6.6%/a compounded monthly, and Nadia negotiates the term of the loan to be 25 years.

- **a)** What will be each monthly payment?
- **b)** After paying for 3 years, Nadia receives an inheritance and makes a one-time payment of $15 000 against the outstanding balance of the loan. How much earlier can she pay off the loan because of this payment?
- **c)** How much will she save in interest charges by making the $15 000 payment?

5. **K** Lisa and Karl are deciding to invest $750 per month for the next 7 years.

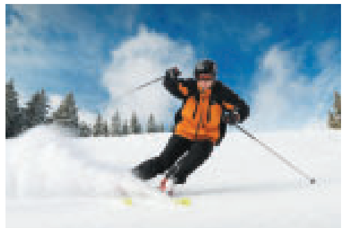
- Bank A has offered them 6.6%/a compounded monthly.
- Bank B has offered them 7.8%/a compounded monthly.

How much more will they end up with by choosing the second offer?

6. Mario decides to pay $250 per month at 5%/a compounded monthly to pay off a $25 000 loan. After 2 years, he is making a bit more money and decides to increase the monthly payment. If he pays $50 extra per month at the end of each 2-year period, how long will it take him to pay off the loan?

7. Natalie borrows \$150 000 at 4.2%/a compounded monthly for a period of 20 years to start a business. She is guaranteed that interest rate for 5 years and makes monthly payments of \$924.86. After 5 years, she renegotiates her loan, but interest rates have gone up to 7.5%/a compounded monthly.
   a) If Natalie would like to have the loan paid off after the original 20-year period, what should her new monthly payment be?
   b) If she keeps her payments the same, how much extra time will it take her to pay off the loan?

8. Peter buys a ski vacation package priced at \$2754. He pays \$350 down and finances the balance at \$147 per month for $1\frac{1}{2}$ years. Determine the annual interest rate, compounded monthly, being charged. Round your answer to two decimal places.



9. **a)** Suppose you have a loan where the interest rate doubles. If you want to keep the same amortization period, should you double the payment? Justify your reasoning with examples.
   T
   b) Suppose you are borrowing money. If you decide to double the amount borrowed, should you double the payment if you want to keep the same amortization period? Justify your reasoning with examples.

10. Laurie borrows \$50 000 for 10 years at 6.6%/a compounded monthly. How much sooner can she pay off the loan if she doubles the monthly payment after 4 years?

11. What are the advantages and disadvantages of using each technology to solve financial problems?
   C
   - a spreadsheet
   - a graphing calculator

Extending

12. A music store will finance the purchase of a rare guitar at 3.6%/a compounded monthly over 5 years, but offers a \$250 reduction if the payment is cash. If you can get a loan from a bank at 4.8%/a compounded annually, how much would the guitar have to sell for to make it worthwhile to take out the loan?

13. The interest on all mortgages is charged semi-annually. You are given a choice of monthly, semi-monthly, bi-weekly, and weekly payments. Suppose you have a mortgage at 8%/a, the monthly payments are \$1000, and the amortization period is 20 years. Investigate the effect on the time to pay off the mortgage if you made each of these payments.
   a) \$500 semi-monthly
   b) \$500 bi-weekly
   c) \$250 weekly

14. Steve decides to pay \$150 per month to pay off a \$6800 loan. In the beginning, the interest rate is 13%/a compounded monthly. The bank guarantees the interest rate for one year at a time. The rate for the next year is determined by the going rate at the time. Assuming that each year the rate drops by 0.5%/a, how long will it take Steve to pay off his loan?

8 Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 8.4, Examples 1 to 4.
- Try Chapter Review Questions 11 and 12.

Q: How do you determine the future value of an annuity?

A1: An annuity is a series of payments or investments made at regular intervals. The future value of an annuity is the sum of all regular payments and interest earned. You can determine the future value of each payment or investment by using the formula $A = P(1 + i)^n$.

Since an annuity consists of regular payments, the future values of the investments, starting from the last, will be P, $P(1 + i)$, $P(1 + i)^2$, ... These form a geometric sequence with common ratio $1 + i$. So the future value of all of the investments is the geometric series $P + P(1 + i) + P(1 + i)^2 + \ldots$, which can be calculated with the formula for the sum of a geometric series.

A2: You can use technology such as a spreadsheet or the TVM Solver on a graphing calculator to calculate the future value of an annuity.

EXAMPLE

The spreadsheet below is set up for an annuity in which 40 regular investments of $250 are made at the end of each compounding period. The annuity earns 2% interest per compounding period.

Since the last $250 investment was deposited at the end of the term, it earned no interest. The first $250 investment earned interest 39 times, but didn't earn interest during the first compounding period because it was deposited at the end of that period.

	A	B	C
1	Number of Compounding Periods Invested	Amount Invested	Future Value
2	0	$250.00	"=B2"
3	"=A2+1"	$250.00	"=B3*(1+0.02)"
4	"=A3+1"	$250.00	"=B4*(1+0.02)*2"

	A	B	C
1	Number of Compounding Periods Invested	Amount Invested	Future Value
2	0	$250.00	$250.00
3	1	$250.00	$255.00
4	2	$250.00	$260.10
5	3	$250.00	$265.30
6	4	$250.00	$270.61
7	5	$250.00	$276.02
8	6	$250.00	$281.54
37	35	$250.00	$499.97
38	36	$250.00	$509.97
39	37	$250.00	$520.17
40	38	$250.00	$530.57
41	39	$250.00	$541.19
42			$15 100.50

The future value of this annuity is $15 100.50

A3: Use the formula for the future value of an annuity, $FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$, where FV is the future value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

Q: How can you determine the present value of an annuity?

A1: The present value of an annuity is the amount of money you have to invest to get a specific amount some time in the future. You can determine the present value of each investment by using the formula $PV = A(1 + i)^{-n}$.

Since an annuity consists of regular payments, the present values of the investments, starting from the first, will be $A(1 + i)^{-1}$, $A(1 + i)^{-2}$, $A(1 + i)^{-3}$, ... These form a geometric sequence with common ratio $(1 + i)^{-1}$. So the present value of all of the investments is the geometric series $A(1 + i)^{-1} + A(1 + i)^{-2} + A(1 + i)^{-3} + \ldots$, which can be calculated with the formula for the sum of a geometric series.

A2: You can use technology such as a spreadsheet or the TVM Solver on a graphing calculator to calculate the present value of an annuity.

EXAMPLE

The spreadsheet below is set up for an annuity earning 0.5% interest per compounding period and providing 20 regular payments of $50.

	A	B	C
1	Number of Compounding Periods Invested	Payment	Present Value
2	1	$50.00	"=B2/1.005"
3	"=A2+1"	$50.00	"=B3/(1.005)^A3"
4	"=A3+1"	$50.00	"=B4/(1.005)^A4"

	A	B	C
1	Number of Compounding Periods Invested	Payment	Present Value
2	1	$50.00	$49.75
3	2	$50.00	$49.50
4	3	$50.00	$49.26
5	4	$50.00	$49.01
6	5	$50.00	$48.77
7	6	$50.00	$48.53
17	16	$50.00	$46.17
18	17	$50.00	$45.94
19	18	$50.00	$45.71
20	19	$50.00	$45.48
21	20	$50.00	$45.25
22			$949.37

The present value of all of the investements in this annuity is $949.37.

A3: Use the formula for the present value of an annuity,

$PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$, where PV is the present value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

Study Aid

- See Lesson 8.5, Examples 1, 2, and 3.
- Try Chapter Review Questions 13 to 17.

PRACTICE Questions

Lesson 8.1

1. Each situation represents an investment earning simple interest. Calculate the interest earned and the total amount.

	Principal	Rate of Simple Interest per Year	Time
a)	$3 500	6%	10 years
b)	$15 000	11%	3 years
c)	$280	3.2%	34 months
d)	$850	29%	100 weeks
e)	$21 000	7.3%	42 days

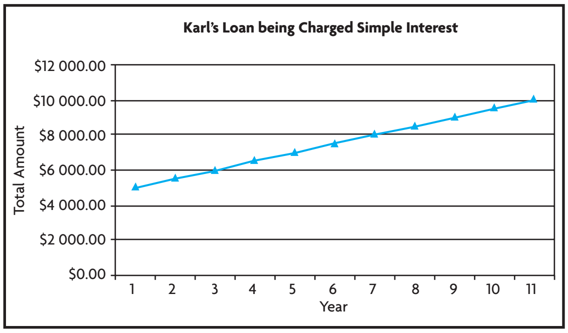
2. Pia invests $2500 in an account that earns simple interest. At the end of each month, she earns $11.25 in interest.

a) What annual rate of simple interest is Pia earning? Round your answer to two decimal places.

b) How much money will be in her account after 7 years?

c) How long will it take for her money to double?

3. Karl borrows some money and is charged simple interest. The graph below shows how the amount he owes grows over time.



a) How much did Karl borrow?

b) What annual interest rate is he being charged?

c) How long will it take before he owes $20 000?

Lesson 8.2

4. Isabelle invests $4350 at 7.6%/a compounded quarterly. How long will it take for her investment to grow to $10 000?

5. Each situation represents a loan being charged compound interest. Calculate the total amount and the interest being charged.

	Principal	Rate of Compound Interest per Year	Compounding Period	Time
a)	$4 300	9.1%	annually	8 years
b)	$500	10.4%	semi-annually	11.5 years
c)	$25 000	6.4%	quarterly	3 years
d)	$307	27.6%	monthly	2.5 years

6. Deana invests some money that earns interest compounded annually. At the end of the first year, she earns $400 in interest. At the end of the second year, she earns $432 in interest.

a) What interest rate, compounded annually, is Deana earning? Round your answer to two decimal places.

b) How much did she invest?

7. Vlad purchased some furniture for his apartment. The total cost was $2942.37. He paid $850 down and financed the rest for 18 months. At the end of the finance period, Vlad owed $2147.48. What annual interest rate, compounded monthly, was he being charged? Round your answer to two decimal places.

Lesson 8.3

8. Calculate the present value of each investment.

	Rate of Compound Interest per Year	Compounding Period	Time	Future Value
a)	6.7%	annually	5 years	$8 000
b)	8.8%	semi-annually	2.5 years	$1 280
c)	5.6%	quarterly	8 years	$100 000
d)	24.6%	monthly	1.5 years	$850

9. Roberto financed a purchase at 9.6%/a compounded monthly for 2.5 years. At the end of the financing period, he still owed $847.53. How much money did Roberto borrow?

10. Marisa invests $1650 for 3 years, at which time her investment is worth $2262.70. What interest rate, compounded annually, would yield the same results? Round your answer to two decimal places.

Lesson 8.4

11. For each annuity, calculate the future value and the interest earned.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	$2500 per year	7.6%	annually	12 years
b)	$500 every 6 months	7.2%	semi-annually	9.5 years
c)	$2500 per quarter	4.3%	quarterly	3 years

12. Naomi wants to save $100 000, so she makes quarterly payments of $1500 into an account that earns 4.4%/a compounded quarterly. How long will it take her to reach her goal?

Lesson 8.5

13. Ernie wants to invest some money each month at 9%/a compounded monthly for 6 years. At the end of that time, he would like to have $25 000. How much money does he have to put away each month?

14. For each loan, calculate the amount of the loan and the interest being charged.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	$450 per year	5.1%	annually	6 years
b)	$2375 every 6 months	9.2%	semi-annually	4.5 years
c)	$185.73 per quarter	12.8%	quarterly	3.5 years
d)	$105.27 per month	19.2%	monthly	1.5 years

15. Paul borrows $136 000. He agrees to make monthly payments for the next 20 years. The interest rate being charged is 6.6%/a compounded monthly.

a) How much will Paul have to pay each month?

b) How much interest is he being charged over the term of the loan?

16. Eden finances a purchase of $611.03 by making monthly payments of $26.17 for $2\frac{1}{2}$ years. What annual interest rate, compounded monthly, is she being charged? Round your answer to two decimal places.

17. Chantal purchases a moped for $1875.47 with $650 down. She finances the balance at 6.6%/a compounded monthly over 4 years. How much will Chantal have to pay each month?

Lesson 8.6

18. Starting at age 20, Ken invests $100 per month in an account that earns 5.4%/a compounded monthly. Starting at age 37, his twin brother, Adam, starts saving money in an account that pays 7.2%/a compounded monthly. How much more money will Adam need to invest each month if he wants his investment to be worth the same as Ken's by the time they are 55 years old?

19. Jenny starts a business and borrows $100 000 at 4.2%/a compounded monthly. She can afford to make payments between $1000 and $1500 per month. How much sooner can she pay off the loan if she pays the maximum amount compared with the minimum amount?

20. Kevin purchases a guitar on a payment plan of $17.85 per week for $2\frac{1}{2}$ years at 13%/a compounded weekly. What was the selling price of the guitar?

8 Chapter Self-Test

Loan #1	
Year	**Amount Owed**
1	$3796
2	$3942
3	$4088

Loan #2	
Year	**Amount Owed**
2	$977.53
3	$1036.18
4	$1098.35

1. For each investment, determine the total amount and the interest earned.

	Principal	Rate of Interest per Year	Time
a)	$850	9% simple interest	6 years
b)	$5460	8.4% compounded semi-annually	13 years
c)	$230 per month	4.8% compounded monthly	$6\frac{1}{2}$ years

2. The amounts owed for two different loans are shown at the left.
   a) For each loan, determine whether simple interest or compound interest is being charged. Justify your answer.
   b) What annual interest rate is each loan being charged? Round your answer to two decimal places.
   c) How much was each loan originally?
   d) Determine the future value of each loan after 10 years.

3. Betsy inherits $15 000 and would like to put some of it away for a down payment on a house in 8 years. If she would like to have $25 000 for the down payment, how much of her inheritance must she invest at 9.2%/a compounded quarterly?

4. Derek invests $250 per month for $6\frac{1}{2}$ years at 4.8%/a compounded monthly. How much will his investment be worth at the end of the $6\frac{1}{2}$ years?

5. Simone wants to save money for her retirement. Her two best options are 5.88%/a compounded monthly or 6%/a compounded annually. Which option should she choose? Why?

6. Anand's parents have been paying $450 per month into a retirement fund for the last 30 years. The fund is now worth $450 000. What annual interest rate, compounded monthly, are Anand's parents earning? Round your answer to two decimal places.

7. Yvette wants to invest some money under these conditions:
- Each quarter for the next 17 years, she wants to earn 8.4%/a compounded quarterly.
- After 17 years, she plans to reinvest the money at 7.2%/a compounded monthly.
- She wants to withdraw $5000 per month for the 10 years after the initial 17 years.

How much more would she have to invest per quarter if she earned 7.2%/a compounded quarterly for the first 17 years and 8.4%/a compounded monthly for the next 10 years?

8 Chapter Task

Saving for Retirement

Steve, Carol, and Lisa get their first full-time jobs and talk about saving for retirement. They are each 22 years old and plan to work until they are 55.

Steve starts investing immediately and puts aside $150 per month. Carol wants to enjoy life a bit and decides to start contributing when she is 30. Lisa thinks that they are both starting too early and decides to wait until she is 42 before starting to save.

Assume that Steve, Carol, and Lisa are each earning 9%/a compounded monthly. Carol and Lisa want to accumulate the same amount as Steve upon retirement. When they retire, Steve wants his investment to last 10 years, Carol wants hers to last 15 years, and Lisa wants hers to last 20 years.

? How much will Steve, Carol, and Lisa be able to withdraw monthly upon retirement?

A. What strategies will you use to solve this problem? Justify your strategies.

B. How much money will Steve have accumulated by the time he is 55?

C. For how many months will Carol and Lisa be making payments?

D. How much will Carol and Lisa have to put away each month to meet their goals?

E. For how many months will each person withdraw money?

F. How much will each person be able to withdraw from his or her nest egg each month?

Task *Checklist*

- ✔ Did you explain and justify your strategies?
- ✔ Did you show your work?
- ✔ Did you support your calculations with appropriate reasoning?
- ✔ Did you explain your thinking clearly?

Chapters **7–8** Cumulative Review

Multiple Choice

1. Determine S_{21} for the series $2.8 + 3.2 + 3.6 + 4.0 + ...$

a) 142.8 **c)** 10.8
b) 104 **d)** 142.4

2. Identify the sequence that is not geometric.

a) 4, 16, 64, 256, …
b) 30, 6, 1.2, 0.24, …
c) 2, 6, 7, 21, 22, …
d) 5, 5, 5, 5, …

3. Consider the sequence $1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, ...$ Determine t_8.

a) $\frac{128}{2187}$ **c)** $-\frac{128}{2187}$
b) $\frac{64}{79}$ **d)** $-\frac{64}{79}$

4. The first three terms of the sequence 8, *a*, *b*, 36 form an arithmetic sequence, but the last three terms form a geometric sequence. Determine all possible values of *a* and *b*.

a) $(a, b) = (1, -6)$ **c)** $(a, b) = (16, 24)$
b) $(a, b) = (-1, 6)$ **d)** $(a, b) = (12, 24)$

5. The fifth term of a geometric series is 405 and the sixth term is 1215. Calculate the sum of the first nine terms.

a) 147 615 **c)** 49 205
b) 8100 **d)** 36 705

6. Determine the first six terms of the sequence defined by $t_1 = -5$ and $t_n = -3t_{n-1} + 8$.

a) $-5, 23, -61, 191, -565, 1703$
b) $-5, 26, -70, 202, -598, 1786$
c) $-5, 23, 61, 191, 565, 1703$
d) $-5, 9, -51, 129, -411, 1209$

7. Choose the correct simplified expansion for the binomial $(x - 3)^5$.

a) $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x - 243$
b) $x^4 - 15x^3 + 90x^2 - 270x + 405$
c) $x^6 - 15x^5 + 90x^4 - 270x^3 + 405x^2 - 243x$
d) $x^5 - 15x^4 + 90x^3 - 270x^2 + 405x$

8. After 15 days, 90% of a radioactive material has decayed. What is the half-life of the material?

a) 1.45 days **c)** 7.5 days
b) 4.52 days **d)** 11.45 days

9. Determine the annual interest rate, compounded annually, that would result in an investment doubling in seven years.

a) 10.4% **c)** 7%
b) 14% **d)** 11.45%

10. How long will it take for $5000 invested at 6%/a compounded monthly to grow to $6546.42?

a) 4.5 years **c)** 40 months
b) 3 years **d)** 48 months

11. Marisa has just won a contest. She must decide between two prize options.

- Collect a lump-sum payment of $50 000.
- Receive $800 at the end of every quarter for 10 years from an investment.

The investment earns 8%/a, compounded quarterly. How much more money would she have if she chooses the lump sum?

a) $3009.89 **c)** $348.92
b) $1678.41 **d)** $30.99

12. An annuity written as a geometric series has *r* equal to 1.005. Determine the annual interest rate for the annuity if the interest is compounded monthly.

a) 12% **c)** 5%
b) 0.5% **d)** 6%

13. Lee wants to buy a plasma television. The selling price is $1894. The finance plan includes $150 down with payments of $113 at the end of each month for $1\frac{1}{2}$ years. Determine the annual interest rate being charged, if the interest is compounded monthly.

a) 3.25% **c)** 20.06%
b) 1% **d)** 24%

14. Mr. Los is planning to buy a sailboat. He decides to deposit \$300 at the end of each month into an account that earns 6%/a interest, compounded monthly. At the end of four years, he uses the balance in the account as a down payment on a \$56 000 sailboat. He gets financing for the balance at a rate of 8%/a, compounded monthly. He can afford payments of \$525 per month. If interest rates remain constant, how long will it take him to repay the loan?

a) 10 years and 6 months
b) 12 years and 9 months
c) 9 years
d) 8 years and 10 months

15. Which option, if any, would allow you to repay a loan in less time?

a) decrease the regular payment
b) increase the regular payment and decrease the interest rate
c) decrease the regular payment and increase the interest rate
d) none of the above

16. Determine which best describes the regular payment on an amortized loan.

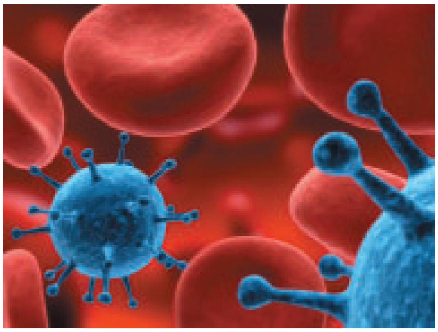
a) the average of all interest payments
b) the fixed periodic payment made up of interest and principal
c) the average of all principal payments
d) the payment of principal only

Investigations

17. Medicine Dosage

Marcus has a bacterial infection and must take 350 mg of medication every 6 h. By the time he takes his next dose, 32% of the medication remains in his body.

a) Determine a recursive formula that models this situation.
b) What will the amount of medication in his body level off to?
c) How long will it take for the medication to reach this level?



18. Financial Planner

You are a financial planner with a new client. Mr. Cowan, who just turned 37, is celebrating his son's fourth birthday.

As his financial planner, he asks you to develop a financial plan for him. Mr. Cowan wants to set up an education fund for his son, Bart, by depositing \$25 at the end of each month until Bart turns 18. The fund earns interest at 6%/a, compounded monthly.

a) Show why the sequence of the monthly amounts in the fund is a geometric sequence. Determine an expression for t_n, the value of the first deposit after n months.
b) How many payments will take place by the time Bart turns 18? Determine the balance in the fund on Bart's 18th birthday.
c) Create a spreadsheet to represent the monthly balance in the fund. Use it to verify your answer in part (b).
d) How much would there be in the fund if Mr. Cowan deposits \$50 per month instead of \$25?