3.7 Factoring a Sum or Difference of Cubes

Math Learning Target:
"I can factor fully a Sum **or** Difference of Cubes." $\Delta^3 \pm \square^3$

Ex.1 Apply the Factor Theorem to factor completely:

a) $y^3 + 8 = P(y)$

$\because P(-2)=0$

$\therefore y+2$ is a factor

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & 0R \end{array}$$ ← expected

$\therefore y^3+8 = (y+2)(y^2-2y+4)$

$a=1\ b=-2\ c=4$

b^2-4ac

$=(-2)^2-4(1)(4)$

$=4-16$

$=-12$

$\therefore b^2-4ac<0$

$\therefore$ no real roots

$\therefore P(x)$ is factored completely

b) $8x^3 - 27 = P(x)$

Hint:
$P(\frac{3}{2}) = 8(\frac{3}{2})^3 - 27$

$= 8(\frac{27}{8})-27$

$= 27 - 27$

$\because P(\frac{3}{2})=0$

$\therefore x-\frac{3}{2}$ is a factor

$$\begin{array}{r|rrrr} \frac{3}{2} & 8 & 0 & 0 & -27 \\ & \downarrow & 12 & 18 & 27 \\ \hline & 8 & 12 & 18 & 0R \end{array}$$

$\therefore P(x)=(x-\frac{3}{2})(8x^2+12x+18)$

$8x^3-27=(x-\frac{3}{2})2(4x^2+6x+9)$

$=(2x-3)(4x^2+6x+9)$

$b^2-4ac<0$

c) $a^3 - b^3 = P(a)$

$\because P(b)=0$

$\therefore a-b$ is a factor

$$\begin{array}{r|rrrr} b & 1 & 0 & 0 & -b^3 \\ & \downarrow & b & b^2 & b^3 \\ \hline & 1 & b & b^2 & 0 \end{array}$$ Remainder

$\therefore a^3-b^3=(a-b)(a^2+ba+b^2)$

$=(a-b)(a^2+ab+b^2)$

Factor Formula for a Difference of Cubes:

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Factor Formula for a Sum of Cubes:

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Let's verify the result by expanding:

$$(a - b)(a^2 + ab + b^2)$$

$$= a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3$$

$$= a^3 - b^3$$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Ex.2 Use the appropriate "new" formula to factor completely:

a) $8x^3 - 27$ (from the previous slide)

$= (2x)^3 - (3)^3 \quad \therefore a = 2x, b = 3$

$= (2x-3)\left((2x)^2 + (2x)(3) + (3)^2\right)$

$= (2x-3)(4x^2 + 6x + 9)$

b) $27x^3 + 125y^3$

$= (3x)^3 + (5y)^3$

$= (3x+5y)(9x^2 - 15xy + 25y^2)$

The Factor Theorem can be applied to any expression.
However, it ***may*** be more difficult to use than if one recognizes the expression as a sum/difference of cubes.
Hence, the following algorithm is suggested, from now on, when required to factor:

STOP Is the expression a sum/difference of cubes?
If so, use the appropriate formula.
Otherwise, apply the Factor Theorem directly.

Entertainment: p.182 #2acegi, 3, 4acegi, 5ac, 6

Are you factoring fully?

From yesterday's Entertainment:

Complete p.177 #6de, 7e, 8e, 9, 10, 14, 16, 17

p.177

8 Factor fully. Use the factored form of f(x) to sketch the graph of each function.

e) $f(x) = x^3 - x^2 + x - 1$

$= x^2(x-1) + 1(x-1)$

$= (x-1)(x^2+1)$

if $f(x)=0$

$x-1=0$ or $x^2+1=0$

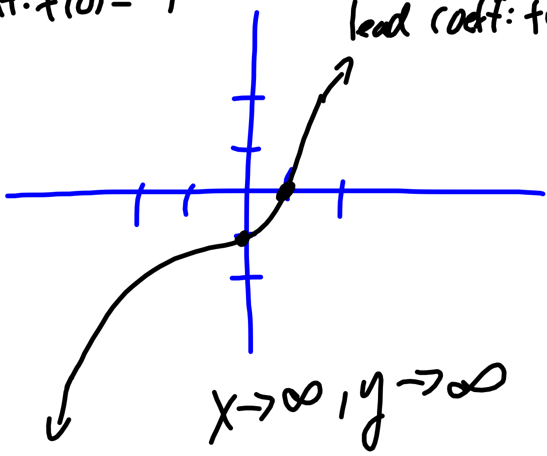
$x=1$ $\quad$ $x^2=-1$

No Real Solution

y-int: $f(0)=-1$

cubic

lead coeff: +ve



$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

9. The polynomial $12x^3 + kx^2 - x - 6$ has $2x - 1$ as one of its factors. Determine the value of k.

$x-3$

$2x-1$ is a factor

$x - \frac{1}{2}$

$\therefore P(\frac{1}{2}) = 0$

$\therefore P(\frac{1}{2}) = 12x^3 + kx^2 - x - 6$

$0 = 12(\frac{1}{2})^3 + k(\frac{1}{2})^2 - (\frac{1}{2}) - 6$

$= 12(\frac{1}{8}) + \frac{1}{4}k - \frac{1}{2} - 6$

$= \frac{3}{2} + \frac{k}{4} - \frac{1}{2} - 6$

$-\frac{k}{4} = \frac{2}{2} - 6$

$\frac{k}{4} = 5$

$k = 20$

p.177

10. When $ax^3 - x^2 + 2x + b$ is divided by $x - 1$, the remainder is 10. When it is divided by $x - 2$, the remainder is 51. Find a and b.

$\therefore P(1) = 10$ and $P(2) = 51$

$a(1)^3 - (1)^2 + 2(1) + b = 10$

$a - 1 + 2 + b = 10$

$a + b = 9$

$\therefore 6 + b = 9$

$b = 3$

Linear System

$a(2)^3 - (2)^2 + 2(2) + b = 51$

$8a - 4 + 4 + b = 51$

$$\begin{array}{r} 8a + b = 51 \\ -a - b = -9 \\ \hline 7a \quad = 42 \end{array}$$

$a = 6$

14. Show that $x - a$ is a factor of $x^4 - a^4$. $= P(x)$

$P(a) = (a)^4 - a^4$

$= 0$

$\because P(a) = 0 \therefore x - a$ is a factor.

Also,

$P(x) = x^4 - a^4$

$= (x^2 - a^2)(x^2 + a^2)$

$= (x-a)(x+a)(x^2 + a^2)$

p.177

16. Use the factor theorem to prove that $x^2 - x - 2$ is a factor of $x^3 - 6x^2 + 3x + 10$.

$P(x) =$

$= (x-2)(x+1)$

$\therefore x = 2$ or $x = -1$

$\therefore$ check $P(2) = 0$ and $P(-1) = 0$

$P(2) = (2)^3 - 6(2)^2 + 3(2) + 10$
$= 8 - 24 + 6 + 10$
$= 0$
$\therefore x-2$ is a factor

$P(-1) = (-1)^3 - 6(-1)^2 + 3(-1) + 10$
$= -1 - 6 - 3 + 10$
$= 0$
$\therefore x+1$ is a factor

$\because (x-2)$ and $(x+1)$ are factors of $P(x)$
$\therefore x^2 - x - 2$ is also a factor of $P(x)$

17. Prove that $x + a$ is a factor of $(x + a)^5 + (x + c)^5 + (a - c)^5$.