First FORMATIVE 6.2

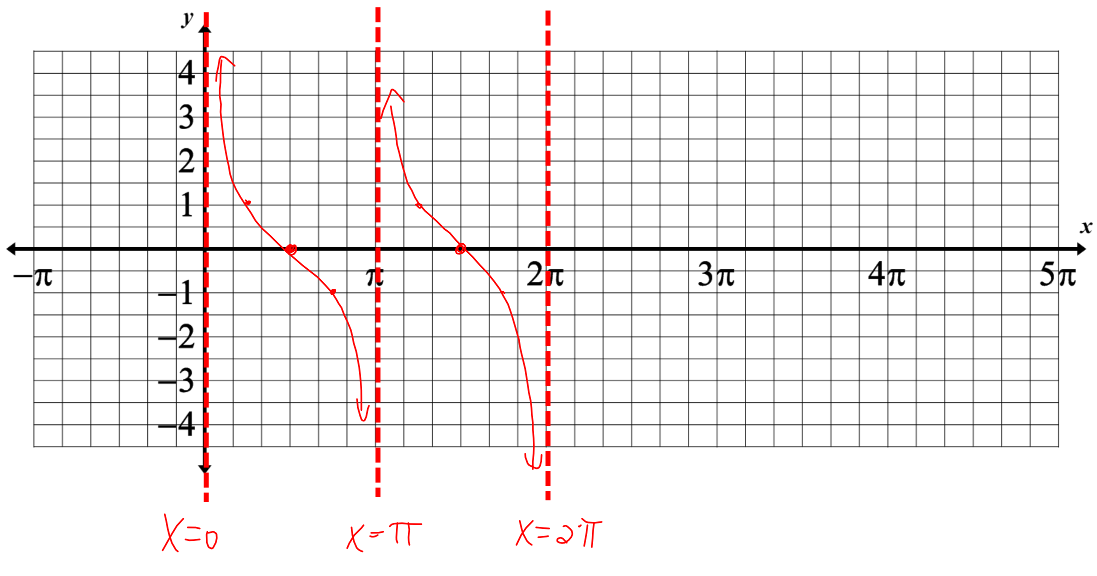
Last Day's Work: **READ** p.352
GRAPH (do not sketch): p.353 #1, 2, 3, 7*; *for #7 graph on the interval $-2\pi \leq x \leq 2\pi$

p.353 **3.** Find an expression that describes the location of each of the following values for $y = \cot x$, where $n \in I$ and x is in radians.

a) vertical asymptotes
b) x-intercepts

b) $\{x \in \mathbb{R} \mid x = \frac{\pi}{2} + n\pi, n \in I\}$

$y = \cot x$



6.6 Modelling with Trigonometric Functions

Math Learning Target:

"I can model and solve problems that involve trigonometric functions and radian measure."

Let's work through the examples in the text on pp.354-360

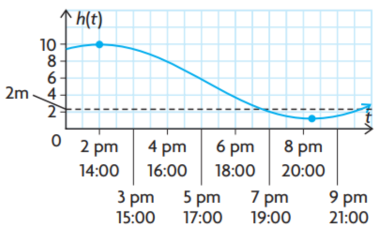
Ex.1: The tides at Cape Capstan, New Brunswick, change the depth of the water in the harbour. On one day in October, the tides have a high point of approximately 10 m at 2 p.m. and a low point of approximately 1.2 m at 8:15 p.m. A particular sailboat has a *draft* of 2 m. This means it can only move in water that is at least 2 m deep. The captain of the sailboat plans to exit the harbour at 6:30 p.m.

Can the captain exit the harbour safely in the sailboat at 6 p.m.?

EXAMPLE 1 Modelling the problem using a sinusoidal equation

Create a sinusoidal function to model the problem, and use it to determine whether the sailboat can exit the harbour safely at 6 p.m.

Solution



A sinusoidal function can be used to model the height of the water versus time. Draw a sketch to get an idea of when the captain needs to leave. It appears that the captain will have enough depth at 6:30 p.m., but you cannot be sure from a rough sketch.

$$H(t) = a\cos(k(t-d)) + c$$

$$a = \frac{10-1.2}{2}$$

$$a = 4.4$$

Choose the cosine function to model the problem, since the graph starts at a maximum value. The amplitude, period, horizontal translation, and equation of the axis need to be determined.

Use the maximum and minimum measurements of the tides to calculate the amplitude of the function. This gives the value of a in the equation.

$$\text{Period} = \frac{2\pi}{k} \quad \Rightarrow k = \frac{2\pi}{\text{period}}$$

$$12.5 = \frac{2\pi}{k}$$

$$12.5k = 2\pi$$

$$k = \frac{2\pi}{12.5} = \frac{4\pi}{25}$$

In a sinusoidal function, the horizontal distance between the maximum and minimum points represents half of one cycle.

Since a maximum tide and a minimum tide occur 6 h 15 min apart, the period must be 12.5 h. The period can be used to determine the value of k in the equation.

$$c = \frac{10+1.2}{2}$$

$$c = 5.6$$

The equation of the axis is the mean of the maximum and minimum points. This can be used to determine the value of c in the equation.

A function that models the tides at Cape Capstan is

$$H(t) = 4.4\cos\left(\frac{4\pi}{25}(t-2)\right) + 5.6.$$

The parent cosine function starts at a maximum point.
If we let $t = 0$ represent noon, then our function needs a maximum at $t = 2$ (or 2 p.m.). We use a horizontal translation right 2 units. Therefore $d = 2$.

fix typo

$$H(6.5) = 4.4\cos\left(\frac{4\pi}{25}(6.5-2)\right) + 5.6$$

$$= 4.4\cos\left(\frac{18\pi}{25}\right) + 5.6$$

$$\doteq 2.80 \text{ m}$$

To determine the water level at 6:30 p.m., let $t = 6.5$.

Since the depth of the water is greater than 2 m at 6:30 p.m., the sailboat can safely exit the harbour.

EXAMPLE 2 Representing a situation described by data using a sinusoidal equation

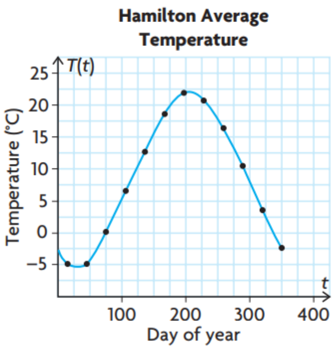
The following table shows the average monthly means of the daily (24 h) temperatures in Hamilton, Ontario. Each month's average temperature is represented by the day in the middle of the month.

Month	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
Day of Year	15	45	75	106	136	167	197	228	259	289	320	350
°C	−4.8	−4.8	−0.2	6.6	12.7	18.6	21.9	20.7	16.4	10.5	3.6	−2.3

a) Plot the temperature data for Hamilton, and fit a sinusoidal curve to the points.
b) Estimate the average daily temperature in Hamilton on the 200th day of the year.

Solution A: Using the data and reasoning about the characteristics of the graph

a)



Plot the data, and sketch a smooth curve through the points.

The curve appears to be sinusoidal, so use $y = a\sin(k(t-d)) + c$ as the model for this situation.

$$a = \frac{\text{maximum} - \text{minimum}}{2}$$

$$a = \frac{21.9 - (-4.8)}{2}$$

$$a = 13.35$$

$$c = \frac{\text{maximum} + \text{minimum}}{2}$$

$$c = \frac{21.9 + (-4.8)}{2}$$

$$c = 8.55$$

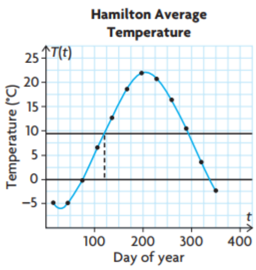
Estimate the maximum and minimum temperatures for the year from the graph. Use these temperatures to calculate the values of a and c. The value of a gives the amplitude. The sine function has been stretched vertically by a factor of 13.35.

The value of c gives the horizontal axis. The sine function has been vertically translated by 8.55 units on the Temperature axis. Lightly draw a horizontal line through your graph at this value.

$$\text{Period} = \frac{2\pi}{k}, \text{ so } k = \frac{2\pi}{\text{period}}$$

$$k = \frac{2\pi}{365}$$

The value of k in the equation is determined by the period. Assume that the cycle repeats itself every year (365 days).



To determine the value of d, estimate where the horizontal axis first intersects the curve.

Since this graph appears to have been translated to the right, $d \doteq 116$.

$$T(t) = 13.35\sin\left(\frac{2\pi}{365}(t-116)\right) + 8.55$$

Replace the parameters in the general sine equation.

b) $T(t) = 13.35\sin\left(\frac{2\pi}{365}(t-116)\right) + 8.55$

$$T(200) = 13.35\sin\left(\frac{2\pi}{365}(200-116)\right) + 8.55$$

$$\doteq 21.8\ °\text{C}$$

Let $t = 200$, and evaluate the sine function.

This model predicts that the average daily temperature in Hamilton on the 200th day of the year is about 21.8 °C.

Since sinusoidal functions are periodic, they can be used (where appropriate) to make educated predictions.

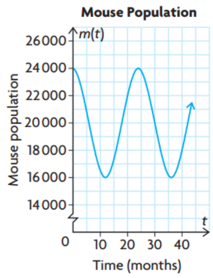
EXAMPLE 3 Analyzing a situation that involves sinusoidal models

The population size, O, of owls (predators) in a certain region can be modelled by the function $O(t) = 1000 + 100 \sin\left(\frac{\pi t}{12}\right)$, where t represents the time in months and $t = 0$ represents January. The population size, m, of mice (prey) in the same region is given by the function $m(t) = 20\,000 + 4000 \cos\left(\frac{\pi t}{12}\right)$.

a) Sketch the graphs of these functions.
b) Compare the graphs, and discuss the relationships between the two populations.
c) How does the mice-to-owls ratio change over time?
d) When is there the most food per owl? When is it safest for the mice?

Solution

a) Graph the prey function.



$m(t) = 4000 \cos\left(\frac{\pi t}{12}\right) + 20\,000.$

The mouse population has a maximum of 24 000 and a minimum of 16 000.
$a = 4000$
The amplitude of the curve is 4000.

$c = 20\,000$
The axis is the line $m(t) = 20\,000$.

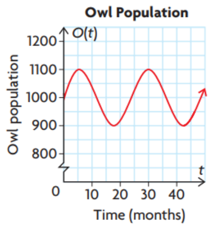
$k = \frac{\pi}{12}$, so the period $= \frac{2\pi}{k}$

$$\text{period} = \frac{2\pi}{\frac{\pi}{12}}$$

$$\text{period} = 2\pi \times \frac{12}{\pi} = 24$$

The period is 24 months.

Graph the predator function.



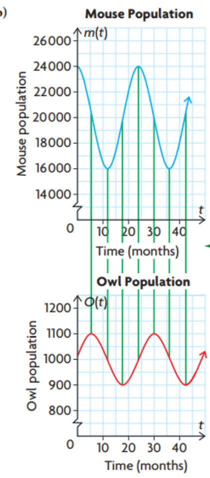
$O(t) = 100 \sin\left(\frac{\pi t}{12}\right) + 1000.$

The owl population has a maximum of 1100 and a minimum of 900.
$a = 100$
The amplitude of the curve is 100.

$c = 1000$
The axis is the line $O(t) = 1000$.

$k = \frac{\pi}{12}$ as above, so this period is also 24 months.

b)

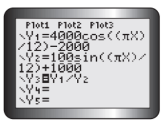


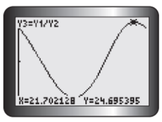
The graphs can be compared, since the same scale was used on both horizontal axes. As the owl population begins to increase, the mouse population begins to decrease. The mouse population continues to decrease, and this has an impact on the owl population, since its food supply dwindles. The owl population peaks and then also starts to decrease. The mouse population reaches a minimum and begins to rise as there are fewer owls to eat the mice. As the mouse population increases, food becomes more plentiful for the owls. So their population begins to rise again. Since both graphs have the same period, this pattern repeats every 24 months.

c) The following table shows the ratio of mice to owls at key points in the first four years.

Time	Mice	Owls	Mice-to-Owl Ratio
0	24 000	1000	24
6	20 000	1100	18.2
12	16 000	1000	16
18	20 000	900	22.2
24	24 000	1000	24

There seems to be a pattern. Enter the mouse function into Y1 of the equation editor of a graphing calculator, and enter the owl function into Y2. Turn off each function, and enter Y3 = Y1/Y2.





The resulting graph is shown. The ratio of mice to owls is also sinusoidal.

d) The most food per owl occurs when the ratio of mice to owls is the highest (there are more mice per owl).

The safest time for the mice occurs at the same time, when the ratio of mice to owls is the highest (there are fewer owls per mouse).

This occurs near the end of the 21st month of the two-year cycle.

In Summary

Key Ideas

- The graphs of $y = \sin x$ and $y = \cos x$ can model periodic phenomena when they are transformed to fit a given situation. The transformed functions are of the form $y = a \sin (k(x - d)) + c$ and $y = a \cos (k(x - d)) + c$, where
  - $|a|$ is the amplitude and $a = \frac{\text{max} - \text{min}}{2}$
  - $|k|$ is the number of cycles in 2π radians, when the period $= \frac{2\pi}{k}$
  - d gives the horizontal translation
  - c is the vertical translation and $y = c$ is the horizontal axis

Need to Know

- Tables of values, graphs, and equations of sinusoidal functions can be used as mathematical models when solving problems. Determining the equation of the appropriate sine or cosine function from the data or graph provided is the most efficient strategy, however, since accurate calculations can be made using the equation.

Discuss 2-part Summative

Entertainment pp. 360-362 #1, 3, 5, 7, 8, 10*, 11%, 13

*Answers in the text are incorrect. Change them to:

10a) $n(t) = 3.7\cos(\frac{2\pi}{365}(t - 172)) + 12$

10b) 9.2 hours

Also, for #10a change the instruction from "*n*th" day to "*t*-th day"

% remember Example 2? Also, the answer for 11 is incorrect. The "d" value is not 116; it is approx. 102

Discuss 2-part Summative

Given: $0 \leq \theta \leq 2\pi$

$\sin\theta = \frac{1}{2}$ $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$